GROUPS GENERATED BY TWO OPERATORS SATISFYING TWO CONDITIONS.

BY PROFESSOR G. A. MILLER.

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THE largest group G generated by two operators s_1, s_2 satisfying two conditions of the form

$$s_1^{\alpha_1} s_2^{\alpha_2} \cdots = 1, \quad s_1^{\beta_1} s_2^{\beta_2} \cdots = 1$$

is, in general, of infinite order. For some special values of the exponents $\alpha_1, \alpha_2, \cdots; \beta_1, \beta_2, \cdots$ the order of G is necessarily finite,* but the general necessary and sufficient condition that the order of G be finite does not seem to be known. We proceed to a consideration of all the possible cases where a series of consecutive exponents is composed of numbers which have the common value unity while all of the other exponents are equal to zero, as these cases appear fundamental in the general problem.

§1. *Consecutive Exponents Equal to Unity While all the Others are Zero.*

It is easy to verify that all these possible cases are included in the following five general forms:

$$(s_1 s_2)^\alpha = (s_2 s_1)^\beta = 1, \quad (s_1 s_2)^\alpha = (s_1 s_2)^\beta s_1 = 1,$$
$$(s_1 s_2)^\alpha = (s_2 s_1)^\beta s_2 = 1, \quad (s_1 s_2)^\alpha s_1 = (s_1 s_2)^\beta s_1 = 1,$$
$$(s_1 s_2)^\alpha s_1 = (s_2 s_1)^\beta s_2 = 1.$$

As the first two conditions are equivalent to the single condition $(s_1 s_2)^\delta = 1$, δ being the highest common factor of α and β, the order of G is infinite in this case.† When $\alpha = \beta$ the fourth pair of conditions reduces also to a single condition and hence the order of G must also be infinite in this special case. We proceed to prove that G must be a cyclic group of finite order in each of the other possible cases.

To prove this fact we may first observe that G must be cyclic

**American Journal of Mathematics*, vol. 31 (1909), p. 167.
† *Ibid.*

whenever s_1, s_2 satisfy any one of the given five pairs of conditions except the first, since the cyclic group generated by $s_1 s_2$ includes the operators s_1, s_2 in each of these cases. When either the second or the third pair of conditions is satisfied, the order of $s_1 s_2$ is explicitly a divisor of α and hence the order of G is a divisor of α. That the order of G is exactly α in each of these cases results directly from the fact that it is possible to find two operators in any cyclic group of order α such that they satisfy the conditions imposed upon s_1, s_2. This result may be expressed as follows:

If two operators s_1, s_2 are such that identity can be obtained by taking them alternately as factors, both when we have an even number and also when we have an odd number of such factors, then they generate a cyclic group whose order divides one-half of this even number, and every cyclic group may be generated by two operators satisfying two such conditions.

To simplify the considerations as regards the two remaining pairs of conditions it may be well to observe the following elementary theorem: If α and β are any two numbers and if G is an arbitrary cyclic group it is always possible to find a pair of generating operators of G such that they satisfy the condition $t_1^\alpha = t_2^\beta$.* When the fourth pair of the given conditions is satisfied it results directly that the order of $s_1 s_2$ is a divisor of $|\alpha - \beta|$. When $\alpha = \beta$ the order of G is infinite, as was observed above. When $\alpha \neq \beta$ G is clearly a cyclic group of order $|\alpha - \beta|$, since it is possible to find two generators of such a cyclic group which satisfy the conditions imposed on s_1, s_2 by these conditions.

It remains to consider the last one of the five given pairs of conditions. Since s_1 and s_2 are commutative it results that these conditions may also be expressed as follows:

$$s_1^{\alpha+1} s_2^{\alpha} = s_1^{\beta} s_2^{\beta+1} = 1.$$

Hence

$$s_1^{\alpha+1} = s_2^{-\alpha}, \quad s_1^{\beta(\alpha+1)} = s_2^{-\alpha\beta} = s_2^{-(\alpha+1)(\beta+1)}.$$

From the last equation it results that $s_2^{\alpha+\beta+1} = s_1^{\alpha+\beta+1} = 1$. Hence the order of G is a divisor of $\alpha + \beta + 1$. Moreover, in any cyclic group whose order is $\alpha + \beta + 1$ it is possible to find two operators which satisfy these conditions and also generate the group. This proves that the order of G is exactly $\alpha + \beta + 1$ whenever s_1, s_2 satisfy this pair of conditions. The preceding considerations constitute a proof of the following theorem:

* *Quarterly Journal of Mathematics*, vol. 36 (1905), p. 51.

*If identity can be obtained in two ways by forming the continued product with two operators s_1, s_2 taken alternately as factors, then the largest group G generated by s_1, s_2 is of finite order except when the number of factors in both products is even, or when the number of these factors is the same odd number in both products and the same operator occurs an odd number of times in each. When the total number of times that each factor appears in the two products is the same and the number of factors in each product is odd then G is a cyclic group whose order is this total number.**

§ 2. *Conditions Imposed upon the Possible Groups by the Single Relation $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α and β being Relatively Prime.*

Since the operators s_1, s_2 are supposed to satisfy the equation

$$(s_1s_2)^\alpha = (s_2s_1)^\beta$$

where α, β are relatively prime, it results immediately from the fact that s_1s_2 and s_2s_1 have the same orders that each of these two operators generates the other. Hence we may assume that

$$s_1s_2 = (s_2s_1)^\gamma.$$

The group G generated by s_1, s_2 involves invariantly the cyclic group generated by s_1s_2 since this cyclic group is generated also by $s_1^{-1} \cdot s_1s_2 \cdot s_1 = s_2s_1$. Hence G involves an invariant cyclic subgroup which gives rise to a cyclic quotient group.

Suppose that the order of s_1s_2 is n. It is evident that n is prime to γ. Moreover, it is possible to select s_1, s_2 so that n is an arbitrary number prime to γ, for a generator of the cyclic group of order n is transformed into its γth power by an operator s_2 whenever γ is prime to n. If we let t represent this generator and $s_2^{-1}t$ represent s_1, we have a set of operators which satisfy the given conditions. The order of s_2 is a multiple of the exponent to which γ belongs modulo n, but it is not restricted in any other way. Hence the order of G is any multiple of n multiplied by this exponent, but it is not subject to any other restriction. This proves the following theorem:

If two operators satisfy the condition $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α and β being relatively prime, the order of s_1s_2 may have any arbitrary value n which is prime to α and β, and hence $s_1s_2 = (s_2s_1)^\gamma$. The order of s_2 is an arbitrary multiple of the exponent e to

* Cf. *American Journal of Mathematics*, vol. 31 (1909), p. 182.

which γ belongs modulo n, and hence the order of G, the group generated by s_1 and s_2, is an arbitrary multiple of en. The group generated by $s_1 s_2$ is invariant under G and leads to a cyclic quotient group and hence all such groups are solvable.

From this theorem it results immediately that if two operators satisfy two conditions of the form

$$(s_1 s_2)^\alpha = (s_2 s_1)^\beta,$$

α, β being relatively prime, they may always generate an arbitrary cyclic group, since it is always possible to assume that the order of s_1 is arbitrary and that $s_1 = s_2^{-1}$. Hence two operators satisfying two such conditions must always generate a solvable group, but these conditions are insufficient to restrict the order of the possible groups. That is, two such conditions may always be satisfied by the two generators of any group in an infinite system of solvable groups of finite order.

§ 3. *Consecutive Exponents Equal to Unity in Both Members of Two Equations.*

At the end of the preceding section we considered the possible groups when the two conditions may be expressed as follows:

$$(s_1 s_2)^{\alpha_1} = (s_2 s_1)^{\beta_1}, \quad (s_1 s_2)^{\alpha_2} = (s_2 s_1)^{\beta_2},$$

where α_1, β_1; α_2, β_2 are two pairs of relatively prime numbers. When these exponents are not relatively prime these conditions are not generally sufficient to restrict the possible groups to solvable groups, and it is evident that whenever each member of the equations involves an even number of operators, the two operators being taken alternately, there can be no upper limit to the order of the possible groups, since any operator and its inverse could be used for s_1, s_2. It remains therefore only to consider the cases where at least one member involves an odd number of such factors.

When one member involves an odd number of such factors while the other involves an even number the equation is evidently equivalent to one of the following:

$$(s_1 s_2)^\alpha s_1 = 1, \quad (s_2 s_1)^\beta s_2 = 1.$$

Hence (s_1, s_2) is the cyclic group generated by $s_1 s_2$. If the second conditional equation also reduced to one of the form considered in section 1, the matter would require no further

consideration. Hence we may assume that the second equation assumes the following form:

$$(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2.$$

Since the preceding condition implies that s_1, s_2 are commutative, this equation becomes

$$s_1^{\alpha_1-\beta_1+1} = s_2^{\beta_1-\alpha_1+1} \quad \text{or} \quad s_1^k = s_2^{-k}.$$

For the same reason the first equation must assume one of the following two forms

$$s_1^l = s_2^{1-l}, \quad s_1^l = s_2^{-1-l}.$$

Hence it results that

$$s_1^{kl} = s_2^{2l-kl} = s_2^{k-kl} \quad \text{or} \quad s_1^{kl} = s_2^{2l-kl} = s_2^{-k-kl}.$$

These conditions fix an upper limit for the order of s_2 and hence also for the order of the cyclic group generated by s_1, s_2, unless $2l$ is equal either to k or to $-k$. That is, *two equations of the forms* $(s_1s_2)^\alpha s_1 = 1$, $(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2$ *define a cyclic group of finite order unless* $2\alpha + 1 = \alpha_1 - \beta_1$; *and two equations of the forms* $(s_2s_1)^\beta s_2 = 1$, $(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2$ *define a cyclic group of finite order unless* $2\beta = \beta_1 - \alpha_1 - 1$. When the special conditions are satisfied these operators generate a cyclic group whose order has no upper limit.

It remains to consider the possible groups when the two given conditions are of the forms

$$(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2, \quad (s_1s_2)^{\alpha_2}s_1 = (s_2s_1)^{\beta_2}s_2.$$

On multiplying the members of the first equation by the inverses of the members of the second, there results the equation

$$(s_1s_2)^{\alpha_1-\alpha_2} = (s_2s_1)^{\beta_1-\beta_2}.$$

If the two numbers $\alpha_1 - \alpha_2$, $\beta_1 - \beta_2$ are relatively prime, this equation implies that the cyclic group generated by s_1s_2 is invariant under (s_1, s_2), as was observed above. Since each of the two conditional equations implies that s_1, s_2 correspond to the same operator in the quotient group of (s_1, s_2) as regards the invariant subgroup generated by s_1s_2, and as s_1 and s_2 must also correspond to inverse operators in this quotient group, it results that this quotient group is of order 2, if it is not

identity. Suppose that n is the order of s_1s_2. As n is prime to both $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ according to our hypothesis, it results that a positive number x_0 less than n can be found so that $(\beta_1 - \beta_2)x_0 \equiv 1 \mod n$. Hence $s_2s_1 = (s_1s_2)^\delta$, $\gamma \equiv (\alpha_1 - \alpha_2)x_0 \mod n$. Since s_1^2 is $(s_1s_2)^\delta$ it is necessary that $\gamma^2 \equiv 1 \mod n$. From this it results that n is a modulus of the following congruences:

$$(\beta_1 - \beta_2)\, x_0 \equiv 1, \quad (\alpha_1 - \alpha_2)^2 x_0^2 \equiv 1.$$

Hence $(\beta_1 - \beta_2)^2 \equiv (\alpha_1 - \alpha_2)^2 \mod n$. The value of n must therefore be a divisor of $|\,(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2\,|$, and as the order of (s_1, s_2) cannot exceed $2n$, this proves that the two conditions under consideration generally restrict the order of the group generated by s_1, s_2. In fact, we have proved the theorem:

The largest group generated by two operators which satisfy the two equations

$$(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2, \quad (s_1s_2)^{\alpha_2}s_1 = (s_2s_1)^{\beta_2}s_2,$$

where $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ are two relatively prime numbers, is solvable and has an order which divides $2|\,(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2\,|$.

When this group is abelian it must be cyclic.* As an illustration of the fact that two such equations may define a cyclic group and also a non-cyclic group of order $2\,|\,(\beta_1-\beta_2)^2-(\alpha_1-\alpha_2)^2\,|$ we may use the following two sets of equations:

$$(s_1s_2)^2s_1 = s_2s_1s_2, \qquad s_1s_2s_1 = (s_2s_1)^3s_2;$$
$$(s_1s_2)^2s_1 = (s_2s_1)^3s_2, \qquad s_1s_2s_1 = s_2s_1s_2.$$

From the first set we obtain $s_1s_2 = (s_2s_1)^{-2} = s_2s_1$, since $(s_1s_2)^3 = 1$; and hence the first equation reduces to $s_1^2 = 1$ while the second becomes $1 = s_1s_2^3$ or $s_1 = s_2^3$. Hence we may assume that s_2 is of order 6 and that (s_1s_2) is the cyclic group of order 6, since such operators clearly satisfy each of the two conditional equations of the first set.

On the other hand, the second set of equations implies that $s_1s_2 = (s_2s_1)^2$, and hence s_1^2 is again unity. As s_1 transforms s_1s_2 into its inverse, it results that (s_1, s_2) is the symmetric group of order 6 in this case. It is also easy to verify that if we assume that s_1, s_2 are two operators of order 2 whose product is of order 3, the given conditional equations will be satisfied. It should not be assumed that the order of s_1s_2 is

* *American Journal of Mathematics*, vol. 31 (1909), p. 182.

always equal to $|(\beta_1-\beta_2)^2-(\alpha_1-\alpha_2)^2|$. In fact, this number is 5 when $(s_1s_2)^3s_1=(s_2s_1)^4s_2$ and $s_1s_2s_1=s_2s_1s_2$ but s_1s_2 in this case is identity, as may easily be verified. All that has been proved is that the order of s_1s_2 must always divide $|(\beta_1-\beta_2)^2-(\alpha_1-\alpha_2)^2|$ when the two numbers $\alpha_1-\alpha_2$, $\beta_1-\beta_2$ are relatively prime. By assigning different values to the exponents α_1, α_2, β_1, β_2 so that $\alpha_1-\alpha_2$, $\beta_1-\beta_2$ are relatively prime we may thus obtain defining relations for an indefinite number of solvable groups of finite order. For instance, it is easy to verify that the two equations

$$(s_1s_2)^4s_1=(s_2s_1)^3s_2, \quad (s_1s_2)^7s_1=(s_2s_1)^5s_2$$

define the group of order 32 which involves a cyclic subgroup of order 16 while the remaining operators transform the operators of this cyclic group into their seventh powers.

§ 4. *The Special Case where* $(s_1s_2)^{\alpha_1}s_1=(s_2s_1)^{\beta_1}s_2$ *and* $(s_1s_2)^{\alpha_1-1}s_1=(s_2s_1)^{\beta_1+1}s_2$.

In this special case it is evident that the second equation may be replaced by the simpler one $s_1s_2=(s_2s_1)^{-1}$. Hence the first equation becomes

$$(s_1s_2)^{\alpha_1}s_1=(s_1s_2)^{-\beta_1}s_2, \quad \text{or} \quad (s_1s_2)^{\alpha_1+\beta_1+1}=s_2^2.$$

Since s_2s_1 is transformed into its inverse by s_2 and its $(\alpha_1+\beta_1+1)$-th power is commutative with s_2 it results that the order of s_1s_2 is a divisor of $2(\alpha_1+\beta_1+1)$. On the other hand, it is easy to see that s_1, s_2 may be so selected as to generate the dicyclic group of order $4(\alpha_1+\beta_1+1)$, and hence the order of s_1s_2 is exactly $2(\alpha_1+\beta_1+1)$ whenever the operators s_1, s_2 are subject only to the two conditional equations expressed in the heading of this section. To prove this fact we may assume that $t=s_1s_2$ is an operator of order $2(\alpha_1+\beta_1+1)$ and that $s_1^{-1}ts_1=t^{-1}$. In the dicyclic group of order $4(\alpha_1+\beta_1+1)$ which involves t we may let s_1 be any operator of order 4 which is not generated by t, and then s_2 may be so selected that $s_1s_2=t$. The operators s_1, s_2 determined in this manner evidently satisfy the conditions imposed on s_1s_2 by the equations expressed in the heading of the present section. Hence the theorem:

If two operators are restricted only by the two equations $(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2$, $(s_1s_2)^{\alpha_1-1}s_1 = (s_2s_1)^{\beta_1+1}s_2$ *they generate the dicyclic group of order* $4(\alpha_1 + \beta_1 + 1)$.

From the symmetry of the equations it results that the same group is generated by s_1, s_2 when they satisfy the two conditions $(s_1s_2)^{\alpha_1}s_1 = (s_2s_1)^{\beta_1}s_2$, $(s_1s_2)^{\alpha_1+1}s_1 = (s_2s_1)^{\beta_1-1}s_2$. The close contact of the present paper with the one entitled "Finite groups which may be defined by two operators satisfying two conditions" * should be noted.

FUNDAMENTAL REGIONS FOR CYCLICAL GROUPS OF LINEAR FRACTIONAL TRANSFORMATIONS ON TWO COMPLEX VARIABLES.

BY PROFESSOR J. W. YOUNG.

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THE purpose of this note is to call attention to a simple method for obtaining fundamental regions for cyclical groups of linear fractional transformations on two complex variables. The simplicity of the method is due to the fact that the determination of a fundamental region for a group of the specified kind is made to depend merely on the construction of such a region for a simply isomorphic group on a single complex variable. The method, moreover, may be readily extended to the case in which the number of variables is n, and to certain restricted types of groups which are not cyclical and not linear.

Let V be any (non-identical) linear fractional transformation on two complex variables, and let Γ be the cyclical group generated by V, i. e., consisting of all transformations $V^n (n = 0, \pm 1, \pm 2, \cdots)$. We interpret V as a collineation in a complex plane, and will consider separately each of the five types to which V may belong according to the configuration of its invariant points and lines.

Suppose first that V is of Type I, i. e., has three and only three invariant points forming the vertices of a triangle ABC. Every transformation of Γ then has the same configuration of invariant elements. The group Γ transforms the points on

* *American Journal of Mathematics*, vol. 31 (1909), p. 167.

each of the sides of the invariant triangle according to a group of projective transformations on a single complex variable, the latter group being isomorphic with Γ. Moreover, it is evident that this isomorphism is certainly not multiple if the transformations on the corresponding side of the triangle are not periodic. Suppose, then, that V produces a transformation which is not elliptic (i. e., whose characteristic cross ratio is not of the form $e^{i\theta}$, θ real) on at least one side of the triangle ABC. Let AB be such a side; the cyclical group on AB is then either hyperbolic (non-involutoric) or loxodromic and is simply isomorphic with Γ. It is well known that any cyclical group of hyperbolic or loxodromic transformations on a single complex variable is properly discontinuous;* a familiar form of a corresponding fundamental region is bounded by two chains about the double points.†

Let R_1 be a fundamental region for the cyclical group on AB, and let R_2 be the class of all points on all lines joining C to points of R_1. R_2 is then a fundamental region for the group Γ if the lines AC and BC are regarded as singular. For if P is any point of the complex plane not on a side of the triangle ABC, the line PC meets AB in a point Q. Since R_1 is a fundamental region for the group on AB, there exists in Γ a transformation which transforms Q into a point Q' of R. This transformation transforms P into a point P' on the line $Q'C$, i. e., into a point of R_2. Further, if R_2 contained two distinct interior points P, P' equivalent under a transformation of Γ, the lines PC, $P'C$, which are necessarily distinct under the hypotheses made, would determine two distinct interior points of R_1 equivalent under the group on AB, which is impossible since R_1 is a fundamental region for the latter group.

* A group G of transformations on the points of a space S_n of n (complex) dimensions is said to be properly discontinuous in S_n, provided there exists in S_n a region R_n of n dimensions such that, (1) if P is any point of S_n not in R_n (with the exception possibly of the points of certain "singular" spreads S_k ($k < n$) in S_n), there exists in G a transformation which transforms P into a point of R_n; and (2) two distinct interior points of R_n are not equivalent under any transformation of G. For a more complete discussion of this notion the reader may consult, e. g., Fubini, Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe, Pisa, 1908, pp. 142 ff.

† If the points of the complex line AB be interpreted in the usual way by the real points of a plane or sphere, the fundamental region in question is bounded by two circles about the double points (cf. Klein-Fricke, Elliptische Modulfunktionen, vol. 1, pp. 187 ff). We have used above the language of complex projective geometry, in which the totality of chains on a complex line correspond to the totality of real circles on the representing plane or sphere. Cf. a paper by the author, The geometry of chains on a complex line, *Annals of Mathematics*, 2d ser., vol. 11 (1909), pp. 33–48.

There remains in the consideration of Type I only the case in which the characteristic cross ratio on each of the sides of the triangle ABC is of the form $e^{i\theta}$, θ real. It is well known that a cyclical group of projectivities on a line generated by a projectivity whose cross ratio is $e^{i\theta}$ is properly discontinuous only if θ is commensurable with π. This is due to the fact that if the ratio $\theta : \pi$ is irrational, the corresponding group contains infinitesimal transformations, owing to the fact that under this hypothesis it is possible to determine two integers m, n such that $|m\theta + n \cdot 2\pi| < \epsilon$ for any positive ϵ. Now it is readily seen that if $e^{i\theta_1}$, $e^{i\theta_2}$, $e^{i\theta_3}$ are the characteristic cross ratios on the sides of the triangle ABC, either θ_1, θ_2, θ_3 are all commensurable with π, or else at least two of them are incommensurable with π. This follows directly from the theorem that the product of the three characteristic cross ratios taken in a proper way is equal to unity. If two of the θ's, say θ_1, θ_2, are incommensurable with π, it follows that Γ contains infinitesimal transformations. For it is then possible to determine three numbers m, n, n' such that the two relations $|m\theta_1 + n \cdot 2\pi| < \epsilon$, $|m\theta_2 + n' \cdot 2\pi| < \epsilon$ hold simultaneously.* The corresponding transformation V^m then clearly transforms any point P into an arbitrarily near point P'.

The only properly discontinuous groups of the type now under consideration are, therefore, those of finite order. If the cyclical group induced by such a Γ on one of the sides of the invariant triangle is of the same order as Γ, the same method as that used above in the hyperbolic or loxodromic cases may evidently be employed to construct a fundamental region for Γ. This method fails, however, if the order of the group on each side of the invariant triangle is less than the order of Γ.†

To obtain a fundamental region in this case we may proceed as follows: Let the order of the cyclical group on AB be m, and suppose that the order of Γ is greater than m. Let R_1 be a fundamental region of the group on AB, and let R_1' be a fundamental region of the group on BC generated by V^m. Let R_2 be the class of all points in which a line joining C to a point of R_1 meets a line joining A to a point of R_1'. The region R_2 is then a fundamental region for Γ. For let P be any point of the complex plane not on a side of the triangle ABC,

* For an elementary proof of this fact, and its generalization to n irrational ratios, cf. Minkowski, Diophantische Approximationen, p. 7.

† This could happen, for example, if the orders of the groups on the sides of ABC were respectively 15, 10, and 6 (since $\frac{1}{15} + \frac{1}{10} - \frac{1}{6} = 0$).

and let the line PC meet AB in Q. There then exists in Γ a transformation V^a which transforms Q into a point Q' of R_1. The point P is transformed by V^a into a point of the line $Q'C$. If P' is not a point of R_2, let the line $P'A$ meet BC in S. There then exists a transformation of the cyclical group generated by V^m, i. e. a transformation V^{mb}, which transforms S into a point S' of R_1'. This transformation will transform P' in to the intersection P'' of $Q'C$ and $S'A$, since V^{mb} leaves the line $Q'C$ invariant. But P'' is by definition a point of R_2. On the other hand, suppose R_2 contained two interior points P, P' which are equivalent under a transformation V^e of Γ. If the lines PC, $P'C$ are distinct, the hypothesis would imply the existence of two distinct interior points of R_1 equivalent under Γ, which contradicts the assumption that R_1 is a fundamental region of the group on AB. If, however, P, P' are collinear with C, the lines PA, $P'A$ are distinct. Since the transformation transforming P into P' now leaves PC invariant, it is of the form V^{km}; but this would imply the existence in R_1' of two points equivalent under a transformation of this form, which again contradicts the assumption regarding R_1'. This completes the proof that R_2 is a fundamental region for Γ.

The points on two of the sides of the invariant triangle have in the above discussion been left out of account; i. e., we have regarded the lines in question as singular. It is clear, however, that if the groups generated on such an invariant line are properly discontinuous, we need only add to R_2 a fundamental region for every such group on an invariant line in order to have a region which is fundamental for all points which are transformed in a properly discontinuous way.

The discussion of the remaining four types now offers no difficulty as the first method described above will readily apply. This method may be described in slightly different language as follows: The transformation V transforms the lines through any invariant point according to a group isomorphic with Γ. If this isomorphism is simple, we choose a *fundamental region of lines* for the group on the invariant point. The region consisting of all the points on these lines is a fundamental region for Γ. If V is of Type II, i. e., has two and only two invariant points, the group on one of these points is parabolic, and hence (since it is not of finite order) is simply isomorphic with Γ. If V is of Type III, i. e., has one and only one invariant point, the group on this point is likewise simply isomorphic with Γ. If

V is of Type IV (a homology) the group induced by Γ on any point of the axis of homology is simply isomorphic with Γ (it may not be properly discontinuous, however, in which case Γ has no fundamental region). Finally, if V is of Type V (an elation), the induced group on any point of the axis distinct from the center is parabolic and simply isomorphic with Γ. The method then suffices to determine a fundamental region for every properly discontinuous cyclical group of linear fractional transformations on two complex variables.

It is readily seen, moreover, that the method may be extended to the case where the number of variables is n. It will serve also to simplify the problem of determining a fundamental region of any properly discontinuous group of transformations on the points of an S_n which leaves a point of S_n invariant, provided the group on the S_{n-k} (for some $k = 1, 2, \cdots, n-1$) through this invariant point is simply isomorphic with the given group. The simplification consists in reducing the problem to the determination of a fundamental region for a simply isomorphic group on a smaller number (namely, k) of variables.

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ON THE RELATIVE DISCRIMINANT OF A CERTAIN KUMMER FIELD.

BY PROFESSOR JACOB WESTLUND.

(Read before the American Mathematical Society, September 7, 1910.)

IN a paper published in the *Transactions of the American Mathematical Society* for October, 1910, I determined the fundamental number or discriminant of the algebraic number field $k(\sqrt[p]{m})$, generated by the real pth root of the positive integer m. Denoting this discriminant by d and setting $m = a_1 a_2^2 \cdots a_{p-1}^{p-1}$, where $a_1 a_2 \cdots a_{p-1}$ is not divisible by the square of a prime, the following result was obtained:

$$(1) \qquad d = (-1)^{(p-1)/2} p^{p-2} (a_1 a_2 \cdots a_{p-1})^{p-1}$$

or

$$(2) \qquad d = (-1)^{(p-1)/2} p^p (a_1 a_2 \cdots a_{p-1})^{p-1},$$

according as $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 or not, where $b = a_1 a_2^2 \cdots a_{p-2}^{p-2}$.

By adjoining $\rho = e^{2\pi i/p}$, p being an odd prime, to this field we obtain the Kummer field $k(\sqrt[p]{m}, \rho)$. The object of the present note is to make use of the results mentioned above in determining the relative discriminant D_r of this Kummer field with respect to the subfield $k(\sqrt[p]{m})$.

Denoting by D the discriminant of $k(\sqrt[p]{m}, \rho)$, we have *

$$D = d^{p-1} N(D_r). \tag{3}$$

Similarly

$$D = d'^{p} N(D'_r), \tag{4}$$

where d' is the discriminant of $k(\rho)$ and D'_r the relative discriminant of $k(\sqrt[p]{m}, \rho)$ with respect to $k(\rho)$. But $d' = (-1)^{(p-1)/2} p^{p-2}$ †, hence

$$D = (-1)^{(p-1)/2} p^{p(p-2)} N(D'_r). \tag{5}$$

We have a general method‡ for determining the relative discriminant of any Kummer field with respect to $k(\rho)$, but usually the method is of very little practical value. In this particular instance, however, we are able to apply the general method. Two cases arise according as m is divisible by p or not.

I. *m not Divisible by p.*

Let c be any prime factor of m and $c^i (i < p)$ the highest power of c contained in m. Then

$$c = P_1 P_2 \cdots P_e,$$

where $P_1, P_2, \cdots, P_e$ are prime ideals in $k(\rho)$ of degree f, and $p - 1 = ef$. Now m contains $P_l^i (l = 1, 2, \cdots, e)$, and since i is prime to p it follows that c^{p-1} is the highest power of c contained in D'_r. Hence

$$D'_r = (a_1 a_2 \cdots a_{p-1})^{p-1} \lambda^n, \tag{6}$$

where $\lambda = (1 - \rho)$.

To determine the exponent n we proceed as follows. Let s be the highest exponent $\leqq p$ for which there exists a number α

* Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 4 (1894–95), p. 206.

† Hilbert, p. 327.

‡ Hilbert, pp. 393–394.

in $k(\rho)$ such that

$$m \equiv \alpha^p \bmod \lambda'.$$

If $m^{p-1} \equiv 1$, mod p^2, then $s = p$. For if $\alpha \equiv m$, mod λ, we have

$$\alpha^p \equiv m^p, \text{ mod } \lambda^p$$

and hence

$$\alpha^p \equiv m, \text{ mod } \lambda^p,$$

since $p = \lambda^{p-1}$. In this case it follows that D'_r is prime to p and hence $n = 0$.

If $m^{p-1} \not\equiv 1$, mod p^2, then $s = p - 1$. For if $s = p$ we should have

$$\alpha^p \equiv m, \text{ mod } \lambda^p,$$

and hence

$$\alpha \equiv m, \text{ mod } \lambda,$$

from which follows that

$$m^p \equiv m, \text{ mod } p^2,$$

which is contrary to the hypothesis. Hence in this case $\lambda^{2(p-1)}$ is the highest power of λ contained in D'_r. Hence $n = 2(p-1)$.

But if $m^{p-1} \equiv 1$, mod p^2, we should have

$$b^p a_{p-1}^{p(p-1)} \equiv b a_{p-1}^{p-1}, \text{ mod } p^2,$$

since $m = b a_{p-1}^{p-1}$, and hence

$$b^{p-1} \equiv a_{p-1}^{p-1}, \text{ mod } p^2.$$

And conversely if $b^{p-1} \equiv a_{p-1}^{p-1}$, mod p^2, it follows that $m^{p-1} \equiv 1$, mod p^2. Hence

$$D'_r = (a_1 a_2 \cdots a_{p-1})^{p-1} \tag{7}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2, and

$$D'_r = (a_1 a_2 \cdots a_{p-1})^{p-1} \lambda^{2(p-1)} \tag{8}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2.

II. *m Divisible by p.*

If c be a prime factor of m different from p, it is evident that c^{p-1} is the highest power of c contained in D'_r. And if p^i is the highest power of p in m, it follows, since i is prime to p,

that λ^{p^2-1} is the highest power of λ contained in D'_r. Hence

$$D'_r = c^{p-1}\lambda^{p^2-1}, \tag{9}$$

where c is the product of the distinct prime factors of m different from p.

From (1), (2), (3), (5), (7), (8), and (9) it then easily follows in both cases that

$$N(D_r) = p^{p-2}. \tag{10}$$

But in the field $k(\sqrt[p]{m})$ we have the following decomposition of p into prime ideal factors, as was proved in the paper mentioned above:

$$p = P^p, \tag{11}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2, and

$$p = P^{p-1}Q, \tag{12}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2, where P and Q are different prime ideals of the first degree.

In the first case we obtain

$$D_r = P^{p-2}. \tag{13}$$

In the second case, however, our method does not enable us to determine the exact powers of P and Q which enter into D_r. We only know that

$$D_r = P^x \cdot Q^y, \tag{14}$$

where $x + y = p - 2$.

PURDUE UNIVERSITY,
October, 1910.

NOTE ON RECIPROCAL FIGURES IN SPACE.

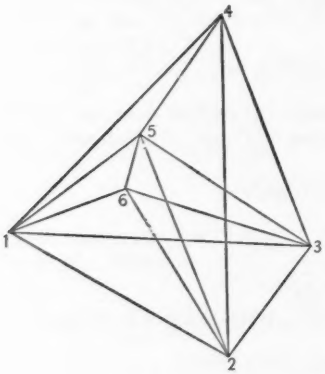
BY PROFESSOR PETER FIELD.

MAXWELL [Collected Works, page 523. Also see Rankine, *Philosophical Magazine* for February, 1864] defines figures in three dimensions as reciprocal when they can be so placed that every line in the one is perpendicular to a plane face of the other and every vertex in the one is represented by a closed polyhedron with plane faces in the other.

The simplest case [Maxwell, loc. cit., page 524] is that of 5

points in space with their 10 connecting lines. This forms a figure composed of 5 tetrahedrons and having 10 triangular faces. The 5 points which are the centers of the spheres circumscribing the 5 tetrahedrons, when connected, form another figure which is reciprocal to the first, and stresses proportional to the areas of the faces of one figure when applied along the edges of the other will keep its vertices in equilibrium. In either one of these figures, 4 of the points might be considered as the vertices of a tetrahedral frame and the fifth point as the point of concurrence of 4 forces which keep the frame in equilibrium. The reciprocal figure then exhibits the magnitudes of the forces and the stresses in the different members of the frame.

Although the idea of reciprocal figures in space is not of much importance in mechanics, because a reciprocal figure can be constructed only in case certain conditions are satisfied, nevertheless it seems of some interest to construct a figure which can be interpreted as representing a frame held in equilibrium by four nonconcurrent forces and for which a reciprocal figure can be constructed.



In the figure at the left, 1, 2, 3, 4 represents a given frame. The line of action of the force at 4 is taken as 45, 5 being any point on the line. As the forces acting at the 4 vertices are in equilibrium, they lie on a hyperboloid [Föppl, Technische Mechanik, volume II, page 158] and it is therefore possible to draw a line through 5 which meets the lines of action of the three remaining forces. Let 6 be any point on this line and join the points as in the figure. The stresses in 15 and 16 can be chosen arbitrarily (as the whole figure regarded as a frame has two more edges than are necessary to make it rigid) and the remaining stresses are then fixed.

The figure can be considered as made up of the tetrahedrons

1234, 1236, 1256, 1356, 2356, 1254, 1354, and 2354. About each of these tetrahedrons a sphere can be circumscribed and the centers of these spheres could be taken as the vertices of a figure which is reciprocal to the given one, i. e., it would be a possible solution. The most general reciprocal figure which can be constructed must contain arbitrary constants corresponding to the fact that the stresses along 15 and 16 can be chosen arbitrarily. This is brought out clearly by constructing the reciprocal figure by the method used by Klein and Wieghardt (*Archiv der Mathematik und Physik*, 3 Reihe, Band 8, Heft 2) which may be summarized as follows:

There are given two sets of rectangular axes that are parallel. Assume a system of cells in the x, y, z space which are so arranged that they fill a certain boundary polyhedron doubly. Assign to each cell C_j a linear function

$$t + \theta_j = \xi_j x + \eta_j y + \zeta_j z.$$

To the cell C_j in the x, y, z space there corresponds a point $P_j \equiv (\xi_j, \eta_j, \zeta_j, \theta_j)$ whose projection in the ξ, η, ζ space has the coordinates ξ_j, η_j, ζ_j. Similarly, to the cell C_k there corresponds the point P_k. If C_k and C_j are neighboring cells, the plane between the two has the equation

$$(\xi_j - \xi_k)x + (\eta_j - \eta_k)y + (\zeta_j - \zeta_k)z - (\theta_j - \theta_k) = 0,$$

which is a plane perpendicular to the line $P_j P_k$. If the origin is taken as a vertex in the reciprocal diagram, $t = 0$ for the corresponding cell. From the equations of the boundary planes it is possible to find the values of t (and consequently the coordinates of the vertices in the reciprocal diagram) for the other cells. In this way the reciprocal figure can be constructed.

In order to prevent the detail work from becoming too cumbersome, take the frame as $P_1 \equiv (0, 0, 0)$, $P_2 \equiv (1, 0, 0)$, $P_3 \equiv (0, 1, 0)$, $P_4 \equiv (0, 0, 1)$, $P_5 \equiv (1, 1, 1)$, and $P_6 \equiv (1, -2, 1)$. The relation between the figure and its reciprocal can then be tabulated as follows:

Cell	Value of t	ξ	η	ζ	θ
1234	0	0	0	0	0
5123	bz	0	0	b	0
5134	bx	b	0	0	0
5124	by	0	b	0	0
6235	$c(x+y)+(b-c)z-c$	c	c	$b-c$	c
6345	$(b-c)x+c(y+z-1)$	$b-c$	c	c	c
6234	$(x+y+z-1)(3c-b)$	$3c-b$	$3c-b$	$3c-b$	$3c-b$
6245	$(b-c)(x+z-1)+cy$	$b-c$	c	$b-c$	$b-c$

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MATHEMATICAL PHYSICS FOR ENGINEERS.

Einführung in die Theorie des Magnetismus. Von R. GANS. Leipzig, Teubner, 1908. vi + 110 pp.

Einführung in die Maxwellsche Theorie der Elektrizität und des Magnetismus. Von CL. SCHAEFER. Leipzig, Teubner, 1908. viii + 174 pp.

Funktionentafeln mit Formeln und Kurven. Von E. JAHNKE und F. EMDE. Leipzig, Teubner, 1909. xii + 176 pp.

PROFESSOR Jahnke is editing a series of texts for engineers. The exact title of the series is Mathematisch-physikalische Schriften für Ingenieure und Studierende. It is published by the firm of B. G. Teubner with the usual typographical excellence of all such works issued by this house. The titles of the volumes which have already appeared, in addition to the three at the head of this review, are Elektromagnetische Ausgleichsvorgänge in Freileitungen und Kabeln, Die Theorie der Besselschen Funktionen,* Die Vektoranalysis und ihre Anwendungen in der theoretischen Physik (2 volumes),† Theorie der Kräftepläne.‡ There are further announced some twenty-five volumes to which titles and authors have already been assigned, and we are assured that a longer continuation of the series is in contemplation. The texts, or should they be called tracts, are all of moderate size, varying from a bit over 100 to somewhat less than 200 pages. From those which have hitherto appeared it is already evident that each has a definite aim and mission to fulfil, which indeed it does fulfil, and that the series as a whole, if not carried so far as to lose its vitality, will form a

* Reviewed in this BULLETIN, volume 16 (1910), p. 385.
† Reviewed in this BULLETIN, volume 17 (1910), p. 100.
‡ Still others have appeared since the date of this review.

valuable collection for students, engineers, and (let us add) mathematicians.

As it is chiefly mathematicians to whom this review is addressed, it will not be amiss to state why this series of tracts seems to us particularly useful to them. In the first place mathematicians, as a general rule in this country at least, are certainly not over-familiar with mathematical physics. The doctorate is given at our universities to many a student of mathematics who has hardly a respectable knowledge of mechanics — to say nothing of the mathematical theories of optics, electromagnetism, thermodynamics, and other physical sciences. The specialist thus turned out may be very highly trained, indeed somewhat fine from over-training, but he is essentially uneducated, essentially narrow. He is unable to appreciate where a great deal of his mathematics arose and what were the physical problems which necessarily called it into being. Moreover, he is really not infrequently as yet unfit to teach with a broad unprejudiced view the elementary calculus, elementary mechanics, and advanced calculus which he may be called upon to teach. Under these circumstances it is but small wonder that many leaders in the various departments of our scientific and engineering schools take an indifferent if not antagonistic attitude toward mathematical instruction as it is. In the second place it is very difficult for the young mathematician who feels the inadequacy of his training upon the practical side to make up his deficiency. The massive works on mathematical physics are written for those accustomed to such subjects and are as difficult and perplexing reading to the mathematician as the modern mathematics is to the physicist. What is needed is a series of tracts, written in a simple explanatory style and each dealing with some well connected and restricted field. Toward mathematical physics the mathematician is really in the position of a Studierender. To us Jahnke's series seems admirably fitted not only to serve the class for whom it is intended but to render great service in broadening such of our young pure mathematicians as would desire to extend their knowledge in the direction which is most likely to add life and value to their teaching and sympathy toward their colleagues in technical departments.

In proceeding to the review of the three particular volumes which are now to be considered we shall adopt the chiastic order

and begin with the last. Jahnke and Emde's table of functions and graphs is a book which is likely to find an active working place in the library of any one who has often to reduce problems to numerical answers. The chief tables are essentially four-place tables. Four-figure accuracy is generally sufficient and should be adopted as the standard of the engineer, mathematician, and physicist. There are some problems which require a closer solution than one which allows an error of one or two hundredths of one per cent; but these are few and should not be put in the foreground. A volume of five-place tables of the functions which Jahnke and Emde tabulate with graphs in 176 pages would be of unwieldy size and serve few purposes for which the present work is not available. The type here selected is large, heavy, and easily read; but a paper not quite so shining would have materially lessened the eye-strain attendant upon long continued use of the table. The cuts and rulings upon which they appear are clear and can easily be read graphically for rough work. Moreover, even when the cuts are not to be used for graphical work, they are very useful in showing the general shape of the function tabulated; in some instances, however, we believe that the graphs could just as well have been omitted.

The first table is of $x \tan x$ and $x^{-1} \tan x$ and is very brief; the values of x are at each tenth (in radians) from 0.0 to 1.5. The authors remark below the graphs that: "In order to obtain intermediate values of these functions it is best to plot off the numbers here given on millimeter paper and to read the desired values from the curves thus obtained. A similar remark applies to all later tables in which the interval is taken too large." We do not understand this; it would seem as though if the interval is too large, a replot from the tables would be far worse than reading the curve as it is; what really is needed is a replot with a large number of intermediate values which are not here given in the table. In fact this particular table seems useless except for very rough work; arithmetic interpolation either forward or backward is of course out of the question. Table II contains the roots of some transcendental equations,

$$\tan x = x, \quad \tan x = \frac{2x}{2 - x^2}, \quad \cos x \cosh x = \pm 1,$$

and others. Not only are the roots tabulated, but the formulas from which they may be computed are often given. Table III

is for the interchange of $a + bi$ and re^{i}. The transformation

$$b/a = \tan\psi, \quad a + bi = a \sec\psi\, e^{i\psi}$$

is made and the values of sec ψ and of ψ are tabulated according to the values of tan ψ. If b and a are small numbers so that division and multiplication are easy, this is the best way to tabulate; but if a, b are any four-figure numbers, it would probably be better to tabulate log sec ψ and ψ. Table IV gives

$$\frac{\pi}{2}x, \quad e^{x}, \quad e^{-x}, \quad \frac{e^{x}}{\sqrt{\tfrac{1}{2}\pi x}}, \quad \frac{e^{-x}}{\sqrt{\tfrac{1}{2}\pi x}}.$$

The values of x run by alternate tenths from 0.0 to 6.0. The interval between the entries for x is so great that here again arithmetic interpolation is out of the question. It will therefore be seen that although the functions are given to four figures, the argument is given to only two and this table is not in any true sense a four-place table. A number of the tables are constructed in this manner; the authors are apparently relying on a replot of the tabulated values and a graphical interpolation from the curves thus obtained. For many purposes this is perhaps sufficient.

Table V is on the hyperbolic functions. Here there are eight pages of formulas giving the definition of the functions and the relation between the different functions, the addition theorems, the functions of multiple angles, the powers of sinh x and cosh x, the relation to the exponential, logarithmic, and circular functions, the derivatives of the functions and integrals which contain them either in the integrand or in the primitive, and also approximations to the functions. The only numerical tabulation is of the gudermannian angle γ, which is to seconds while the argument advances by hundredths from 0.00 to 5.00 and by 0.05 from 5.00 to 8.00. It is therefore apparent that to obtain sinh x and cosh x it is necessary to take tan γ and sec γ out of a trigonometric table. In view of the large amount of room given to the formulas and of the convenience and importance of the functions in numerous practical problems, we believe it would have been well also to tabulate sinh x and cosh x or their logarithms directly to four figures as is done in B. O. Peirce's Short Table of Integrals. At this point let us remark that we believe all students of engineering should be

familiar with the use of tables of hyperbolic functions; in some schools these functions are a regular part of the curriculum, and in no case would it require more than a lesson or two to introduce them.

Tables VI and VII give the important functions

$$\mathrm{Si}\, x = \int_0^x \frac{\sin x}{x}\, dx, \quad \mathrm{Ci}\, x = -\int_x^\infty \frac{\cos x}{x}\, dx \quad (x > 0),$$

$$\mathrm{Ei}\, x = \int_{-\infty}^{-x} \frac{e^{-x}}{x}\, dx = \mathrm{li}\, e^x \quad (x > 0 \text{ and } x < 0),$$

$$\int_0^x \cos \tfrac{1}{2}\pi x^2 dx = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos z}{\sqrt{z}}\, dz,$$

$$\int_0^x \sin \tfrac{1}{2}\pi x^2 dx = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin z}{\sqrt{z}}\, dz.$$

Not only are the functions tabulated; the direct and asymptotic series and several approximate formulas which may be used for the purpose of computing the functions are given. In like manner Table VIII, which gives the Γ-function, its logarithmic derivative and some related functions, contains a large number of the important formulas involving the Γ-function. Table IX is a very complete table of the error function and its first six derivatives. In Table X is found Pearson's function

$$F(r, \nu) = e^{-\frac{1}{2}\pi\nu} \int_0^\pi \sin^r x e^{\nu x} dx,$$

and the logarithms of the first 31 Bernoullian numbers. Apart from their practical services, these tables, VI–X, will be found useful to the teacher for various exercises in advanced calculus when dealing with functions defined by integrals. One of the best things a student can learn is how to compute the numerical value of a given function which occurs in his work; he is not so liable to get lost in his subject nor to worry so much over the rigorous end of it.

The elliptic functions and integrals are treated in Table XI which extends to 24 pages. Numerous introductory formulas are given connecting the functions $F(k, \phi)$, $E(k, \phi)$, $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$, ϑ, ϑ_1, ϑ_2, ϑ_3, p, ζ, σ. The elliptic integrals $F(k, \phi)$ and $E(k, \phi)$ are tabulated at intervals of 5° for $\alpha = \sin^{-1} k$ and of 1° for ϕ.

The complete integrals are tabulated separately and, as α nears 90°, the interval is gradually cut down to 6′. For computation with the aid of the ϑ-functions and especially for all formulas involving q, the values of $\log q^{-1}$ are given at intervals of 5′ for α. Brief tables of the ϑ-functions are also constructed. As for the Weierstrassian functions, $\wp'$, $\wp$, ζ, σ are tabulated in the equianharmonic case

$$^{M}g_2 = 0, \quad g_3 = 1, \quad u = \frac{r\omega_2}{180}, \quad \omega_2 = 1.53995$$

for every unit from $r = 0$ to $r = 240$. Tables for the mutual induction and attraction of two coaxial circular currents complete the set. The only comment we would make is to commend the table giving $\log q^{-1}$. Despite the trigonometri canalogies of the functions *sn*, *cn*, *dn* and despite the beauties of $\wp$, ζ, σ, we believe that the functions which should be emphasized in discussing elliptic functions and integrals are the ϑ-functions. In most practical problems the convergence of the series involving q is so rapid that only a very few terms are required. Moreover, for theoretical purposes it is well to familiarize the student early in his work with integral transcendental functions, which play such an important rôle in analysis and of which the ϑ-functions are among the simplest and most useful examples.

After Table XII on zonal harmonics, the volume closes with Table XIII on the Bessel functions. It is a long drawn-out close of 85 pages — nearly half the volume. About all the formulas that one could want for the practical use of the Bessel functions, including a long array of differential equations solvable in terms of them, are tabulated. The values of $J_n(x)$ for $n = \pm\frac{1}{4}$ and $\pm\frac{1}{2}$, $\pm\frac{3}{2}, \cdots, \pm\frac{13}{2}$ are given; for $n = \pm\frac{1}{4}$ the interval in x is 0.2, for the others it is 1.0 from 0 to 50. Just what is the value of a table of $J_{p+\frac{1}{2}}(x)$ to four significant figures when the interval in x is so great as to give only six points on each complete oscillation of the curve? And are we supposed to plot off these values, make a graph, and interpolate graphically? We do not know. Perhaps it is only for integral values of x that $J_{p+\frac{1}{2}}(x)$ is needed. Tables of $J_0(x)$ and $-J_1(x)$ at intervals of 0.01 from 0.00 to 15.50 are offered; these are real four-place tables. Values of $Y_0(x)$, $-Y_1(x)$, $K_0(x)$, $K_1(x)$, $N_0(x)$, $-N_1(x)$, $J_0(ix)$, $-iJ_1(ix)$, $\frac{1}{2}i\pi H_0^{(1)}(ix)$, $-\frac{1}{2}\pi H_1^{(1)}(ix)$, $J_0(x\sqrt{i})$, $\sqrt{z}J_1(x\sqrt{i})$, $N_0(x\sqrt{i})$, $H_0^{(1)}(x\sqrt{i})$, $H_1^{(1)}(x\sqrt{i})$ are set down at less frequent intervals and over

a more restricted total range; also ber x, bei x, and tables for the self-induction and resistance in a straight circular wire carrying an alternating current. Numerous sets of roots of $J_n(x) = 0$ are also to be found. We note further some tables of $J_n(x)$ for integral values of n from 1 to 24 and for such integral values of x as make $J_n(x) > 10^{-18}$; and there are still other tables. It will thus be seen that the subject of Bessel functions is very throughly covered, perhaps disproportionately so. We should remember, however, that the Bessel functions are very important in many applications of mathematics; apparently they are more important than the elliptic functions. Indeed if one had to choose between some mention of the Bessel functions and equal mention of elliptic functions in a second course in calculus, one should unhesitatingly choose the former, especially if the time were somewhat restricted. It has always seemed somewhat peculiar and unfortunate that B. O. Peirce in extending his Short Table of Integrals in its excellent revised edition should have added some 85 formulas on elliptic functions while having the Bessel function represented only by the definition of $J_n(x)$ when n is integral.

From these extended comments it should appear that we have in Jahnke and Emde's Funktionentafeln an extremely useful collection of formulas and values relating to the most important functions other than elementary, that the chief tables are suited for four-place accuracy, and that there are numerous other tables sufficiently good for graphical or slide-rule work. There can be little doubt that the use of the book will be very general among all workers with mathematics.

To review Schaefer's tract on the introduction to Maxwell's theory without appearing so complete and enthusiastic an admirer of the work as to lessen the value of the review is an impossibility; the work seems to have nothing but excellencies. In fact as soon as it came to our notice we immediately introduced it to our classes and followed it closely from start to finish. We shall merely try here to explain the reason for our enthusiasm. Every one is probably familiar with at least the covers of a number of works on Maxwell's theory; those who have sought familiarity with the interior know full well the difficulties encountered; the works err either toward giving a very gross physical and equally restricted mathematical explanation

of the actions in the ether or toward amassing great arrays of mathematical formulas with too little attention to the physics involved. So far as we know, Maxwell in his original treatise strikes a much happier mean between the extremes than his numerous followers and 'expounders.' Schaefer's text is a sort of minor Maxwell, an exposition eminently suitable for students and engineers — and mathematicians. Like Maxwell, Schaefer appreciates the great advantages derivable from using a vector analysis; but like him, he does not wish to assume the necessary knowledge on the part of his readers. Further, the author decides not to use Green's and Stokes's theorems on divergence and curl, but to develop their equivalents directly on the fields he is considering whenever he finds it necessary to obtain results which the theorems would immediately yield. This course of procedure will undoubtedly be the more satisfactory to the greater number of readers of the work.

Chapter I is on electrostatics. After a few words in the way of describing fundamental facts and giving necessary definitions, the author mentions the fluid hypotheses and Coulomb's law for point charges. He then starts in upon the question of action at a distance versus action in a medium. He shows that the field $\mathbf{E}$ which arises from any distribution of charged points satisfies the equation $\operatorname{div} \mathbf{E} = 0$ except at the points; this he calls the first law of action in the medium. By introducing Gauss's theorem that the induction of $\mathbf{E}$ through a surface is 4π times the total charge within, the further result $\operatorname{div} \mathbf{E} = 4\pi\rho$ is obtained. He next shows that the field due to any set of point charges has a potential so that $\mathbf{E} = -\nabla\phi$ or $\nabla \times \mathbf{E} = 0$ or the line integral of $\mathbf{E}$ about a circuit is zero. This he calls the second law of action in the medium and he derives the relation $\nabla \cdot \nabla\phi = -4\pi\rho$. The peculiar beauty of this development is the clear and artful way in which mathematics founded on the conception of point charges is combined with physics founded on the conception of action in a medium. The idea of a medium is given further emphasis by the construction of a hydrodynamic analogy. After a few applications to condensers the author takes up homogeneous and non-homogeneous dielectrics, discusses the difference between free and true electricity, and gives additional applications to condensers. A short section on the energy of the field closes the chapter; but the section is not so short that the author cannot show how in the simplest case the field theory of energy leads to the same value as the

theory of action at a distance. That he cannot handle the general theorem is due merely to the unavailabilty of Green's theorem. The reader of this first chapter ought easily to obtain the idea that action at a distance and action through a medium are two different points of view in accounting for the same phenomena, that they lead to the same mathematical formulation of the problem, that through their mathematics they are interchangeable, and that, as they are thus far equivalent, the reader may feel free to adopt as definitive either hypothesis that future investigations may establish as preferable.

A brief chapter on magnetostatics constructed on the model of the previous one leads to the discussion of the electric current and its magnetic field in Chapter III. The same general method of development is also used here. The field $\mathbf{H}$ due to a straight current is shown to fulfil the field equation $\nabla \cdot \mathbf{H} = 0$, or better $\nabla \cdot \mathbf{B} = 0$ when μ varies. The law that the circuit integral of $\mathbf{H}$ is proportional to the current through the circuit is discussed for the same simple case. It is then stated that these laws hold for any field. What amounts nearly to a demonstration of Stokes's theorem is then given to establish the equations $4\pi c^{-1}\mathbf{C} = \nabla \times \mathbf{H}$. The equation $\nabla \cdot \mathbf{C}$ which follows as a consequence is explained. The vector potential is introduced and the Biot-Savart law for the mutual action of current elements is deduced. Then follow the Maxwell equations $\epsilon\mathbf{E} + 4\pi\mathbf{C} = c\nabla \times \mathbf{H}$, the discussion of Ohm's and Joule's laws, and a careful treatment of the electrostatic and electromagnetic systems of units. It may be observed that Schaefer uses the mixed system. Chapter IV on induction carries on the work to find the equations $\dot{\mathbf{B}} = -c\nabla \times \mathbf{E}$, to integrate the equations by potentials, to present Poynting's theorem and explain its significance, to deduce the self and mutual induction of circuits, and to treat in considerable detail the phenomena connected with electric oscillations in simple and coupled circuits. Chapter V on electric waves deals with plane waves in isotropic dielectrics, reflection and refraction, (for which the laws are obtained by using the previously established boundary conditions for continuity or discontinuity of the components of $\mathbf{D}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$,) and finally the question of electromagnetic waves in metals. There is a great mass of vital electromagnetic physics in this chapter and it is all presented with the author's usual clearness and care in exposition. The reader can hardly get the wrong or fail to get the right point of view.

If this brief summary of the contents of Schaefer's book can serve to justify so unreserved enthusiasm for the work, we shall be happy; we shall be happier if it can induce others to read the book with interest and study it with care. There can now be no sufficient excuse for lack of familiarity with the elements of Maxwell's great theory; this presentation is too simple, too perfect to offer any difficulties; it neither offers too much nor affords too little for a proper introduction to the subject. Jahnke is to be felicitated on securing such a tract for his series; we only wish it had appeared in English.

Gans's tract on magnetism is also written from the point of view of Maxwell's theory, which as is stated in the preface is the point of view more and more adopted in technical work. There is consequently some duplication between this and the previous volume reviewed. Another point of the preface which is noteworthy is that Gans like Schaefer would be very glad to use vector analysis if he quite dared. The cry for some sort of vector analysis is growing as the waste of space and energy and the danger of an incorrect and insufficiently physical view, which arise when cartesian analysis is used, become more prominent. It is probable that mathematicians at large would do well to familiarize themselves with the elements of vector analysis for the purpose of seeing how the little that is needed so badly can best be worked into the courses in mathematics. The plan of giving vectors as a separate and somewhat elaborate course in a few institutions or even in many institutions is not a success; it does not meet the need. Not long ago one of our colleagues, not himself a mathematician, expressed the conviction that all of our engineering students who carry their mathematics into the third year should probably have some vector analysis and that the electrical engineers should certainly have it. In this we entirely concur. And we surely believe that the vectors should come in little by little throughout the student's course in mathematics; the time could readily be saved from analytic geometry, especially from solid analytic geometry. The mere notion and notation of the scalar product of two vectors will of itself work wonders in clarifying a large number of formulas of plane and solid geometry; without the scalar product the ideas of line and surface integrals and of Stokes's and Green's theorems are far from clear. The matter needs serious consideration and needs it now. We, as mathematicians,

must not continue indefinitely to hamper the instruction (alas, too rare) in theoretical physics by our negligence toward a very fruitful and suggestive field of our own science.

Gans starts Chapter I off with the statement that, in the neighborhood of an electric current, space has a certain condition called magnetic! He then proceeds systematically to develop this idea into the Maxwell equations curl $\mathbf{H} = 4\pi\mathbf{C}$. The second chapter treats induction in para- and diamagnetic bodies and results in the Maxwell equations curl $\mathbf{E} = -\dot{\mathbf{B}}$. It may be noted that Gans is using a different set of units from Schaefer. The magnetic potential is introduced and applied to several simple systems of currents. The boundary conditions in $\mathbf{B}$ and $\mathbf{H}$ are established and applied to numerous problems of a practical technical nature. The sphere, shell, prolate ellipsoid in a constant field are also treated. As this requires the introduction of curvilinear coordinates and Laplace's equation, it is clear that Gans is not quite so chary of his mathematics as Schaefer. But is it not a curious state of affairs and quite derogatory to the acumen and common sense and honesty of teachers of mathematics to think that curvilinear coordinates can find their place where vectors are afraid to enter? We go on teaching the same old things in the same old way, except that we may pay relatively more attention to rigor and relatively less to what is useful than in the old days. For instance, take a look at the last edition of Serret-Harnack-Scheffers's Differential- und Integralrechnung and see how it compares with Serret's original. See whether the new material and the amplifications have not to do with limits, convergence, and uniformity rather than with line and surface integrals, Gauss's, Green's, Stokes's theorems, and potential integrals.

Chapter III takes up ferromagnetic bodies. The start is made from the experimental curves of B to H and μ to H. The hysteresis diagram is discussed. Finally the equations

$$\nabla\cdot\mathbf{H} = 4\pi\rho, \quad \mathbf{H} = -\nabla\phi, \quad \nabla\cdot\nabla\phi = -4\pi\rho_0, \quad \phi = \int\int\int \frac{\rho_0}{r}\, dv$$

lead to the conception of the inverse-square law as applied to magnetism. If one looks real sharply he will find as a subheading in § 31 the mention of a magnetic pole! It is quite obvious that Gans is following out the Maxwell theory as indicated in the first sentences of the first chapter and that he is

not giving way in the slightest to the old point of view of magnetic poles and the law of the inverse square. This consistent attitude of the author is highly to be praised. In Chapter IV magnetic energy and force are treated with the aid of the Poynting vector. It is along toward the middle of this the last chapter in the tract that Gauss's methods of determining the magnetic moment of a magnet and the horizontal component of the earth's field are given. An especially commendable point is the analysis which leads up to the refutation of the common but false definition of para- and diamagnetic bodies by the position they assume relative to the lines of a magnetic field. Some mention of self- and mutual induction, Lenz's rule, and the ballistic galvanometer forms a close to the work.

It is indeed good to read a book like this of Gans where magnetism is treated systematically from the point of view of electromagnetism. The ordinary treatises on electromagnetism slip over the special field of magnetostatics with very little attention, the ordinary treatises on magnetism scarcely mention electromagnetism as other than a means and often they found electromagnetic effects on the study of magnetic shells. Now it may seem somewhat inconsistent in us to praise so heartily Schaefer's neat interrelating of action at a distance with action through a medium and still to support with almost equal ardor Gans's complete capitulation to the theory of a medium. But there is a vital difference between electrostatics and magnetostatics in that negative and positive electricity can be isolated, that is, $\operatorname{div} \mathbf{D} \neq 0$, whereas a true magnetic pole is a fiction, that is, $\operatorname{div} \mathbf{B} \equiv 0$. We recommend Gans to our readers as heartily as we recommended Schaefer; but don't begin with Gans. It is certainly true that Jahnke has made an auspicious start with his series of tracts; let us hope that he can keep up on the present high level; short, snappy, scientific, modern tracts are just what we need.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *August*, 1910.

SHORTER NOTICES.

The Fundamental Laws of Addition and Multiplication in Elementary Algebra. By EDWARD V. HUNTINGTON, Harvard University. Reprinted from the *Annals of Mathematics*, Second Series, Volume 8, No. 1 (October, 1906). Publication Office of Harvard University. 44 pp.

THE object of this paper, as stated in the introduction, is "to present a list of fundamental propositions for algebra, from which, on the one hand, all the other propositions of algebra can be deduced, and in which, on the other hand, no superfluous items are included, — a list, in short, which is sufficient and free from redundancies."

The propositions are stated in terms of abstract undefined symbols a, b, etc., and two combinations of these symbols denoted by $\oplus$ and $\odot$. More briefly stated, the question which the paper answers is: "Given a class of elements with two rules of combination, what conditions must such a system satisfy in order to be formally equivalent to one of the systems of ordinary algebra?"

The first conditions imposed are the ten laws:

A_1. $a \oplus b$ is a unique element of the system.
A_2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
A_3. (1) If $a \oplus x = a \oplus y$ then $x = y$.
(2) If $x \oplus a = y \oplus a$ then $x = y$.
A_4. If $\mu x = \mu y$ then $x = y$, μ being an ordinary integer.
A_5. $[a \oplus b = b \oplus a]$.
M_1. $a \odot b$ is a unique element of the system.
M_2. $(a \odot b) \odot c = a \odot (b \odot c)$.
M_3. (1) If $a \odot x = a \odot y$ and $a \oplus a \neq a$ then $x = y$.
(2) If $x \odot a = y \odot a$ and $a \oplus a \neq a$ then $x = y$.
M_4. (1) $a \odot (b \odot c) = (a \odot b) \oplus (a \odot c)$.
(2) $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)$.
M_5. $a \odot b = b \odot a$.

Emphasis is placed on the purely abstract character of these symbols and on the formal character of the deduction from them. "How can we be sure that our deduction is rigorous? . . . The only way to avoid the danger of using in our reasoning other properties besides those expressly stated in the funda-

mental assumptions is to regard them not as axiomatic propositions about numbers, but as *blank forms* in which the letters a, b, c, etc. may denote any objects we please and the symbols $\oplus$ and $\odot$ any rules of combination; · · · The deduction from such blanks must necessarily be purely formal, and hence will not be affected by the troublesome connotations which would be sure to attach themselves to any concrete interpretation of the symbols." This we recognize of course as the general program of modern work in the foundations of mathematics.

The independence or non-redundancy of the system is exhibited in the usual manner, by showing concrete systems in which all but one of the assumptions are verified. Thus A_2 is shown to be independent of the other nine laws by the following systems: "Let the class considered be the class of all rational numbers. Let $a \oplus b = 2(a + b)$ and $a \odot b = ab$."

The following argument is given to show that A_5 is a consequence of the remaining nine of the ten fundamental postulates:

$$(a + b)(c + c) = (a + b)c + (a + b)c$$
$$= ac + bc + ac + bc \qquad (M4_1, M4_2),$$

also

$$(a + b)(c + c) = a(c + c) + b(c + c)$$
$$= ac + ac + bc + bc \qquad (M4_2\ M4_1).$$

Hence

$$ac + ac + bc + bc = ac + bc + ac + bc$$

and

$$bc + ac = ac + bc \qquad (A3_1\ A3_2).$$

Hence

$$(b + a)c = (a + b)c \text{ and } b + a = a + b \ (M4_2\ M3_2).$$

The consistency of the fundamental laws is shown by giving a concrete example which is assumed to be self-consistent in which all the postulates are satisfied. The example chosen is the Argand diagram representing the complex numbers, in reality a vector field. It should be noted that this by no means proves absolutely the consistency of the postulates, but merely transfers the question to another realm. The consistency of each of the many special algebras developed in the paper follows from the fact that it is a subalgebra of this most general algebra employed.

It is shown that many essentially different systems satisfy these ten postulates. Thus they are all satisfied by the numbers

of ordinary algebra with addition and multiplication retaining their usual meaning, and also by a system in which $a \oplus b = a + b + 1$ and $a \odot b = ab + (a + b)$. This gives rise to a discussion of isomorphism and categoricality. Two systems satisfying these general laws are said to be isomorphic with respect to addition and multiplication when the following conditions are satisfied:

1) the elements of the two systems can be brought into one-to-one correspondence ··· and

2) this correspondence can be set up in such a way that whenever a and b in one class corresponds to a' and b' in the other class, then $a + b$ will correspond to $a' + b'$ and $a \times b$ will correspond to $a' \times b'$.

If a set of postulates is such that any two systems which satisfy it are isomorphic, then the set of postulates is said to be categorical.

While the ten postulates given above do not therefore form a categorical system, special laws may be added to them in various ways to form categorical systems. Thus by adding the postulates

E_1. There is a unit element in the system.

F. There are no elements in the system besides those required by the other postulates.

It is then asserted that the algebra of positive integers is completely determined by the postulates A_1, 2; M_1, 2, 3, 4; E_1; F. In a paper avowedly abstract and purely deductive this seems a somewhat daring statement. It is proved, it is true, that these positive integral elements form a closed system with respect to addition and multiplication, but is this equivalent to showing that the set of postulates just given is categorical according to the definition above? The truth seems to be that we are certain our system is as categorical as some other systems to which we may prove it equivalent. Again, as with the question of consistency, the difficulty has simply been transferred to another realm.

Many other categorical systems are obtained in a manner similar to the above by adding special assumptions to those already given.

It should be remarked that in order to form an independent system each of the ten fundamental postulates except A_1 and M_1 must be regarded as making the existence of the sums and products hypothetical. Thus A_2 must read, if stated fully, "If $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c)$ exist, etc."

The original name for the unproved propositions of a mathematical science was "axiom," — a truth so simple that everyone must assent to it whenever the statement is fully comprehended. In this respect the point of view has changed completely. If a, b, $\oplus$, $\odot$ are purely abstract symbols, then no proposition whatever is evident about them. Hence the word "axiom" with its old connotation is being discarded. The paper under review uses "postulate." Other writers, as Veblen and Young, are using "assumption."

This paper should serve two distinct and very useful purposes. The writer of elementary algebras for college use will have at hand a set of postulates which will serve directly as a basis for much of his work and as a model which we hope may guide his way in making the extensions necessary to characterize the complete algebra. This should render less prevalent in the future the numerous logical incongruities that so often have marred otherwise excellent texts.

The subject with which the paper deals is confessedly abstract but the style is so lucid and the mode of treatment so simple that it should be within the reach of students even in the first years in college. It is the feeling of the reviewer that the reader who is to take his first dip into abstract mathematics cannot very well do better than to read this elegant introduction to the logical foundations of algebra.

N. J. LENNES.

Leçons sur la Théorie de la Croissance. By ÉMILE BOREL. Paris, Gauthier-Villars, 1910. vi + 168 pp.

THIS book is one of the excellent series of monographs on the theory of functions appearing under the general editorial direction of M. Borel. It has grown out of two independent courses of lectures on the theory of increase (croissance) delivered by the author at the University of Paris during the winter semesters of 1907–1908 and 1908–1909 respectively. These two courses have been coordinated and unified by M. Arnaud Denjoy.

A function $f(x)$ is said to be increasing if $f(x') - f(x'')$ is positive when $x' - x''$ is positive. The theory of increase is devoted to the investigation of the rate of change of an increasing function $f(x)$ with respect to x as x approaches infinity.

The author begins his preface by giving expression to his growing conviction that the theory of increase is the essential

basis of the theory of functions. But it seems to him preferable to wait for some years before undertaking a systematic theory of functions in which the theory of increase would serve as an introduction the essential results of which would be constantly called into use. This mode of exposition will bring about a considerable simplification; but it requires the employment of new terms and new notations, and these should be introduced with a great deal of care.

Besides the introduction (13 pages) the book contains five chapters, of which the first three are devoted to the general theory of increase and the last two to its applications.

In Chapter I (19 pages) certain fundamental types of increasing functions are considered and with each is associated a sign which is used to denote its order of increase; for instance, the orders of x^n and e^x are said to be n and ω respectively. Operations of addition and multiplication of orders of increase are defined and their laws are developed.

In Chapter II (9 pages) the consideration of approximate orders of increase is taken up. The function

$$f(x)=x^n\,(\log x)^{2+\sin x},$$

for instance, is said to be of order (n) (read "n parenthesis"). The use of (n) in this case to denote the approximate order of $f(x)$ is justified by the fact that the order of magnitude of $f(x)$ is less than that of $x^{n+\epsilon}$ and greater than that of $x^{n-\epsilon}$, where ϵ is any positive quantity. The laws of addition and multiplication of approximate orders are investigated.

The third chapter (32 pages) is devoted to a study of the changes produced in the order of increase of a function by the integration and differentiation of the function.

Chapter IV (44 pages) on analytical applications falls into three parts. In the first part series with constant positive terms are considered. If such a series diverges, the sum of its first n terms, which increases indefinitely with n, possesses, with respect to n, an order of increase which it is useful to know. If the series is convergent one can propose the problem of determining the order of decrease of the remainder after n terms. Both of these questions are treated. Preparatory to a corresponding investigation for infinite products and as contributing to it, the author in the second part of the chapter develops the fundamental properties of the gamma function.

They are deduced in a very simple way from the preceding general theory. Finally, the third part of the chapter is given over to the study of infinite products and especially to a comparison of the increase of functions with that of their zeros.

The last chapter (51 pages) is devoted to arithmetical applications. Its object is to utilize the theory of increase in the classification of incommensurable numbers. An indication of the contents of the chapter is given in the following list of topics: Relations between incommensurable numbers and increasing functions; continued fractions; approximation by means of rational numbers to certain classes of numbers; numbers of the second order and periodic continued fractions; approximation by means of rational numbers to any numbers whatever; approximation by means of algebraic numbers; case of the number e.

In this little book M. Borel has rendered valuable service by emphasizing in an effective way the fact that the theory of increase has a place of growing importance in the development of the theory of functions.

R. D. CARMICHAEL.

Elemente der Mathematik. By JULES TANNERY. Autorisierte deutsche Ausgabe von P. KLAESS. Leipzig, Teubner, 1909. x + 339 pp. 8 marks.

Two objects dominate the teaching of freshman mathematics: the one, which may not too inaccurately be called the classical, endeavors to give the student a certain proficiency in algebraic analysis or in geometrical logic; the other, which may be called the psychological, endeavors to give him a wide view over the realm of mathematics. The classical has for its educative end skill; the psychological, vision. The classical endeavors to build the structure by finishing one room at a time. The psychological endeavors to build first the framework and then to fill in the details. Whether it is better for the great mass of freshmen, who will have little or no mathematics after that year, to learn well how to work Horner's process or to solve trigonometric equations, or whether there should be preferred a knowledge of "what is a function, its variation, and the representation of its values," or a knowledge of how we determine "tangents, areas, volumes, and the new functions, curves, or surfaces thus arising" — is a question now in

process of solution. To further the new view of what is desirable as it has been planned for French schools, the book under review was written.

The introduction of over one hundred pages clears up the notions of the student acquired in his secondary school work. These are usually vague and half-forgotten, and in most cases in such state that the student "has no desire to continue nor does he see any use for what he has already learned." After this review of first principles, the real advance begins and proceeds in a delightfully graceful manner from the consideration of the important conclusions that may be drawn from elementary identities, into the rise and solution of the quadratic equation; thence into coordinates, functions, limits, derivatives, and integrals. The intuition is constantly appealed to and nowhere is there difficulty for any student to follow the easy grade the course pursues. The last chapter, on astronomy, leaves the student face to face with one of the boundless realms he may enter through the mathematical door.

One can but feel that a student who has had such a course as this will have more genuine love and enthusiasm for continuing his mathematical studies than one who has been drilled in the conventional way. The original (or this translation or, let us hope, an English translation) ought to be in the hands of at least every teacher of elementary mathematics.

JAMES BYRNIE SHAW.

Komplex-Symbolik. Eine Einführung in die analytische Geometrie mehrdimensionaler Raüme. Von ROLAND WEITZENBÖCK. Leipzig, Göschen (Sammlung Schubert, LVII), 1908. 191 pp.

THE real subject of this little volume is line geometry in three and higher dimensions. The treatment however is not according to the old familiar methods but by the use of symbolic notation. The symbolism is founded on that of Clebsch, with which the author states that the reader should be familiar in order to follow this book with ease. The reviewer also wishes to emphasize this need of reading something on symbolic notation before undertaking this book. The symbolism however is sufficiently developed so that one can readily understand it, but a previous knowledge makes the reading more enjoyable. The only other attempt known to the reviewer to treat the subject of line geometry symbolically is that of E. Waelsch ("Zur Invariant-

entheorie der Liniengeometrie," *Sitzungsberichte der k. Akademie der Wissenschaften*, Wien, 1889). The author states that he developed the subject without knowledge of Waelsch's work so that there naturally is divergence of treatment.

In many places the author has not made his meaning clear, and this coupled with frequent typographical errors makes the reading rather difficult. But after one has gone through with it the point of view gained is well worth the trouble.

Two kinds of symbols are defined, viz., ordinary and complex. Ordinary symbols are those which obey the commutative law of multiplication, complex symbols are those which do not. The ordinary coefficients are of the first kind, and the symbols used to represent the line coordinates are of the second kind. Thus for line coordinates we have

$$p_i p_k = p_{ik} = -p_{ki} = -p_k p_i.$$

The quantities p_i and p_k have no meaning except as symbols. After these definitions the general properties of complex symbols are discussed and applied to the linear and quadratic complex in three dimensions and to finding the invariants, covariants, and contravariants of systems of lines.

Then follows the discussion of linear systems of lines in higher dimensions. Here the author has contributed much new material. After reading the discussion of the linear complex in s_4 one appreciates how much more direct and simple is the treatment of the same subject by Castelnuovo. But nevertheless after the symbolism is built up it enables one to see much that might otherwise escape him.

Throughout the book the author has made good use of the idea of defining a line, curve, or in higher dimensions, a plane, etc., by the system of lines which cut it. This has a decided advantage for certain problems, since a single equation then represents the line, curve, etc.

The book closes with an excellent chapter outlining a general symbolic analytic geometry and setting forth some of its advantages.

C. L. E. MOORE.

Analytische Geometrie des Punktpaares, des Kegelschnittes und der Fläche zweiter Ordnung. Zweiter Teilband. Von Dr. OTTO STAUDE. Leipzig and Berlin, Teubner, 1910. iv + 452 pp., with 47 figures.

THIS volume continues the work reviewed in the October (1910) BULLETIN, the page and paragraph numbering being continuous, and the entire work containing one thousand pages. The valuable bibliography and notes relating to the whole work cover sixty pages at the end of the present volume. Since the character of the treatment of the various topics is the same as that in the earlier volumes, little comment seems necessary.

After classifying surfaces of the second class in various ways, the author devotes nearly one hundred pages to a detailed study of plane sections of surfaces of the second order. Circular and equilateral hyperbolic sections receive particular attention. The second part, containing three chapters, is devoted to confocal systems of quadrics, and to general focal properties, both of the surfaces and of their plane sections. In the first chapter, by regarding x, y, z, as homogeneous coordinates of the lines of a sheaf on a point O, instead of ordinary point coordinates in space, the properties of confocal systems of cones are very neatly established. The so-called elliptic coordinates of the lines of a sheaf, and of the points of space are explained and used to some extent in subsequent proofs. The axis complexes of confocal systems are discussed and a large number of theorems relating to confocal systems proven. The second chapter is devoted to the Amiot, the MacCullagh, and the Jacobi focal properties. To any one not familiar with the theorems, this chapter will prove interesting reading. The third chapter on broken focal distances contains a large number of theorems and formulas relating to metrical relations existing between the points of the focal ellipse and the points of the corresponding focal hyperbola, and between the points of two conjugate focal parabolas. From these focal properties, the "thread" construction of the ellipsoid is deduced, and the corresponding property and the construction of the ellipse on the plane are shown analytically to be a special case of this.

The third part consists of two long chapters on surfaces of the second order and of the second class in tetraedral coordinates. In the first chapter surfaces are again classified, conditions for degeneracy stated, plane sections discussed, etc., as previously done in ordinary coordinates. The complex of tangents to a surface and the six generator complexes are studied. The theorem is established that the generators fall into two systems, each system being composed of the lines common to three of the six linear generator complexes. These six com-

plexes are thus divided into two groups of three each. The last chapter is devoted largely to polar properties. The orthogonal linear substitutions corresponding to the real polar tetraedra; the properties of tetraedra determined by four generators, two from each system; and the Pascal and Brianchon theorems in space are other topics in this chapter.

This work with its extensive bibliography, and hundreds upon hundreds of references to the literature on the subject, must represent a prodigious amount of labor and pains on the part of the author. As an encyclopaedic reference book it will undoubtedly be useful, although the index does not seem very satisfactory. For example, twenty or more pages are devoted to a discussion and comparison of the Amiot and MacCullagh focal properties, yet neither of these names appears in the index.

D. D. LEIB.

Festschrift zur Feier des 100 Geburtstages Eduard Kummers mit Briefen an seine Mutter und an Leopold Kronecker. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Leipzig, Teubner, 1910. 103 pp.

IT was very appropriate that the memorial address on Kummer should be delivered by K. Hensel, whose remarkable investigations on algebraic numbers entitle him to speak with authority on Kummer's chief triumph, the creation of ideal numbers. We find here an elementary and entertaining account of Kummer's early interest in Fermat's equation $x^\lambda+y^\lambda+z^\lambda=0$, where λ is a prime, which led him to investigate the numbers $a + b\alpha + \cdots + k\alpha^{\lambda-1}$, where α is a complex λ-th root of unity, and $a, b, \cdots, k$ are integers. It appears on excellent authority that Kummer at an early period supposed that he had a complete proof of the impossibility of Fermat's equation and laid before Dirichlet a manuscript purporting to give such a proof. The latter pointed out to Kummer that, although he had proved that any number $f(\alpha)$ was the product of indecomposable factors, he had assumed that such a factorization was unique, whereas this was not true in general. After years of study, Kummer concluded that this chaotic state of affairs following from the non-uniqueness of factorization was due to the fact that the domain of the numbers $f(\alpha)$ was too small to permit the presence in it of the true prime numbers, and was led to his epoch-making creation of ideal numbers. Without attempting to explain the elaborate machinery of Kummer's

method, so delicate that an expert must handle it with the greatest care, and nowadays chiefly of historical interest in view of the simpler and more general theory of Dedekind, Hensel contented himself in his address with a luminous elementary account of the essential features of Kummer's method as developed for a special set of integers. It should be stated that, while it is preferable to take the newer standpoint as regards the foundation of the theory of ideals, this change of viewpoint has not altered the validity of the rich array of fundamental results obtained by Kummer. The brief sketch on page 30 of Kummer's method of proving the impossibility of Fermat's equation, when λ is a regular prime, applies directly only to the first of the two cases into which Kummer divided the discussion. However, the same use of the class number is employed in the second case and this point is the one being emphasized by Hensel. All admirers of Kummer will take keen pleasure in reading this masterly Gedächtnisrede.

The 56 pages of letters from Kummer to his favorite and most gifted pupil Kronecker are of historic value. They give a first-hand view of the progress step by step made by Kummer in his construction of his own imperishable monument.

L. E. DICKSON.

Interpolationsrechnung. Von T. N. THIELE, Em. Professor der Astronomie an der Kopenhagener Universität, Präsident des Vereins Dänischer Aktuare. Leipzig, Teubner, 1909. xii + 175 pp.

IN this book the theory of interpolation is developed so as to emphasize its role in pure mathematics as well as its place in the calculations of applied mathematics. The book is divided into four chapters. The elementary treatment in the first chapter is based on the general interpolation formula of Newton, written

$$(1)\qquad \begin{aligned} X = A &+ (x-a)[\delta'(a,b) + (x-b)\{\delta''(a,\cdots,c) \\ &+ (x-c)\{\delta'''(a,\cdots,d) + (x-d)\delta''''(x,a,\cdots,d)\}\}], \end{aligned}$$

where $a, b, c, d, \cdots$ are distinct values of the argument, $A, B, C, D, \cdots$ are corresponding tabulated values, and $\delta'(a,b), \delta''(a,\cdots,c), \cdots$ are "divided differences" defined by

$$\delta'(a,b) = \frac{A-B}{a-b},\ \delta''(a,\cdots,c) = \frac{\delta'(a,b)-\delta'(b,c)}{a-c},\ \cdots.$$

A number of theorems are proved concerning divided differences, and it is shown, by numerical applications, that formula (1) is easily usable if the arguments be chosen in certain ways. After expressing the derivative of the nth order d^nX/dx^n in terms of divided differences with repeated arguments, the first application is to develop Taylor's series in terms of divided differences. The next application is made to the solution of numerical equations. The coefficients are expressed as divided differences and it is maintained that the interpolation method is one of the best methods of finding the real roots of numerical equations.

Interpolation by means of infinite series is treated to provide for cases where divided differences of no finite order have the value zero. The possibility that a Newton's interpolation formula, even if convergent, does not represent the function in question, is considered. The author maintains, however, that the functions of the most importance in applied mathematics, fortunately, belong to a class that can be interpolated, at least within certain limitations on the argument. He states the theorem that, for this class of functions, the interpolated values coincide with the function in question if for an infinite number of values of the argument, in a finite interval, there is exact coincidence. It seems that a characterization of functions as belonging to applied mathematics is not definite, in the sense that it is a basis for proving a theorem, and it should be emphasized that the method by which the theorem is established involves questions of convergence.

The first chapter ends with a careful investigation into the representation of a^x by a method of interpolation.

The second chapter is devoted to symbolic representation; the rather burdensome notation is made readable by the presentation of numerous numerical examples.

The third chapter treats the question of interpolation when the function has singularities, and presents graphic interpolation as an auxiliary method. Near the end of the third chapter, interpolation by reciprocal differences is presented. By means of these reciprocal differences, there arises a general method of interpolation that involves continued fractions and that is entirely analogous to the method of Newton. The general formula for interpolation by reciprocal differences is given. It is indicated that the precise applicability of reciprocal differences to interpolation is much more extensive, in a certain sense, than

that of divided differences. This arises from the fact that where the method by divided differences is limited to integral functions, the method of reciprocal differences is limited to the most general form of fractional functions. To be sure, the application of reciprocal differences is also much more difficult than that of divided differences.

In the fourth chapter interpolation for functions of two or more variables is presented in an interesting and useful manner.

The book, as a whole, is a scholarly and readable presentation of the elements of the calculus of finite differences, and should be found of value not only to those interested in the arithmetical application of interpolation, but also to those interested in a theoretical treatment of the subject.

H. L. RIETZ.

Plane Geometry, with Problems and Applications. By H. E. SLAUGHT and N. J. LENNES. Allyn and Bacon, Boston, 1910. vi + 280 pp.

THIS book has several features that distinguish it from the conventional high school text. Among these the most noticeable are the gradual introduction of the severely logical forms, and the introduction, for the purpose of making the subject more attractive, of a large number of applications to geometric forms more or less commonly met with in life.

The book is divided into seven chapters, the first five of which correspond in a general way to Books I to V of Euclid, except that certain of the more difficult theorems, and the subject of incommensurable ratios are deferred to the last two chapters. Chapter I begins as usual with an introduction containing the common definitions. Numerical equality and geometric equality, or congruence, are sharply distinguished. The first propositions and problems are then introduced (pages 14–25) in an informal way. Then a few axioms are stated formally, and a number of theorems are given as "preliminary theorems," some of which are easy consequences, and some of which, for the purposes of the text, are assumed. A general discussion on the nature of a demonstration follows, after which proofs are given in the usual form. Aside from the introduction of the applications and the deferring of the matter indicated above, the content of the first five chapters is about that of the usual book. The algebraic form of the treatment is a decided improvement. One particularly pleasing feature is the willing-

ness of the authors to introduce assumptions, as such, wherever they deem it desirable. At the same time, they state frankly that the set of axioms given is not complete, and that intuitional inferences concerning such subjects as order and continuity are drawn freely.

Chapter VI contains the treatment of variable geometric magnitudes, the subject being treated from the standpoint of the graph, a new feature which should aid in clearing up this subject.

Chapter VII contains general remarks on the nature of axioms, proofs, definitions, etc., from the strictly logical point of view; problems on loci, proofs of the incommensurable cases in the proportion theorems, mean and extreme ratio, area of a triangle in terms of its sides, and a few problems on maxima and minima. Length of arc and area of a circle are defined as limits.

The applications, which occur in large numbers throughout the book, are usually taken from architectural designs and various ornamental designs for decorative purposes, such as tile patterns, parquet floors, grill work, etc. These exercises are for the most part very simple, but by bringing into play a large number of straight lines and circle arcs in a single problem, they are a valuable aid in the development of geometric imagination. These exercises also bring into bold relief the principle of symmetry, which is thereby given the prominence that it deserves. Other applications are made to the problems of finding the distance between two points on opposite sides of a pond, measuring the height of a tree, finding the distance between two inaccessible points, cutting the braces for a roof, etc. All these applications should be a valuable aid in holding the interest of the numerous pupils who care comparatively little for the logical features of the subject. An ordinary class would probably find more than enough material of this kind in the book, but the omission of a part of it would in no way affect the continuity of the subject.

A novel feature is the introduction of the sine of an angle, with a few exercises in its use (pages 136–139).

The book as a whole appears to be a very teachable text. The typographical work is good, and the figures are clearly drawn. As compared with the ordinary text, the book has lost nothing and gained much by the innovations.

F. W. Owens.

Shop Problems in Mathematics. By W. E. BRECKENRIDGE, S. F. MERSEREAU, and C. F. MOORE. Boston, Ginn and Company, 1910. vii + 280 pp.

INTENDED for use in trade schools, the book is divided into two distinct portions, the earlier and major part consisting essentially of problems from the pattern shop, forge, foundry, and machine shop; and the remainder in the presentation of a review of certain parts of elementary mathematics which are involved in the earlier problems. Demonstrations are not presented, results only being desired. A chapter on the slide rule is also included, in which an error in statement is implied which may lead the reader to believe that much of slide rule work is exact. The volume gives the impression of a lack of organized sequence in the arrangement of the material in the individual chapters.

C. F. CRAIG.

The Integrals of Mechanics. By O. C. LESTER. Boston, Ginn and Company, 1909. vi + 67 pp.

THE author has collected many of the integrals found in theoretical mechanics and presents them with very little physical interpretation. After an introductory chapter reviewing the applications of the integral calculus to lengths of arc, area, etc., the subjects of center of gravity, moment of inertia, and ellipsoids of inertia are treated in order. Stress is laid on the presentation of formulas for many of the special cases and on theorems derivable from special examples. The general formulas as well as the special ones are obtained by what are at best only semi-rigorous methods. Although the volume is intended as an introduction to theoretical mechanics, most of the material seems usable as an introduction to applied mechanics.

C. F. CRAIG.

Annuaire du Bureau des Longitudes pour l'An 1911. Paris, Gauthier-Villars.

SEVERAL improvements are noticeable in the issue for the current year. The section on the classification of stellar spectra has been rewritten by M. de Gramont; the elements of the moon and major planets are referred to the epoch 1900 instead of 1850; and M. Schulhof contributes a résumé of the information obtained during the last two years concerning Halley's

comet and the comets of 1909. Tables of variable stars have been dropped because the number has become too great for record in a volume of this kind. Minor improvements have also been made in the sections: geography and statistics, coinage, weights and measures, and meteorology.

The appendices contain an account of the sixteenth meeting of the international geodetic association written by M. Poincaré in his well-known luminous and entertaining style. The eclipse 1912 April 17 is treated by M. G. Bigourdan. The central line passes across France; unfortunately the calculated maximum duration of totality is only six seconds and it is doubtful whether even this small interval will be attained. In certain parts the eclipse will be annular, in others total. Obituary notices of Bouquet de la Grye and Paul Gautier are contributed by MM. Poincaré and Baillaud.

ERNEST W. BROWN.

Leçons élémentaires sur le Calcul des Probabilités. Par R. DE MONTESSUS. Paris, Gauthier-Villars, 1908. vi + 191 pp.

THERE is perhaps not much chance for striking novelties, whether of material or of arrangement, in elementary texts on the theory of probabilities; the general model, especially in France, seems deservedly to be Bertrand's excellent Calcul des Probabilités. The present author has on numerous occasions manifested his interest in various questions concerning probability, and in particular in regard to the proper definition of chance; he does not approve of the scheme of founding the laws of chance upon ignorance as to what may happen and upon the law of sufficient reason, but he believes rather that the laws should be developed out of the experience that in the long run certain things do happen in a particular way for a definite percentage of the total number of ways they may happen. This seems to us, as to the author, somewhat more satisfactory than the older method; it is better to say that in tossing coins heads are found to come half the time and hence the probability of heads is one half, rather than to say that there is no reason why heads should fall rather than tails and hence heads will fall half the time.

Not only in his introductory remarks but throughout the book the author discloses philosophical as well as mathematical tendencies. Moreover, the subjects which he touches are numerous; it is not usual to find mention of insurance and of

speculation in addition to the discussion of various games of chance and other such canonical subjects for treatises on probabilities. As many topics are touched upon, they can naturally not all be more than merely touched, and the author has been good enough to indicate many references where the different subjects may be further pursued. The elementary character, the clear style, the varied topics, the careful references, all combine to make the work useful and thoroughly to be recommended to a wide range of readers who have some, not necessarily much, knowledge of mathematics.

E. B. WILSON.

NOTES.

THE April meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 28–29. At this meeting Professor MAXIME BÔCHER will deliver his Presidential Address, the provisional title of which is: "Charles Sturm's Published and Unpublished Work on Differential and Algebraic Equations." Except for the summer meetings, this will be the first united meeting of the whole Society since 1896. A large attendance is expected from all sections of the country. Titles and abstracts of papers to be presented at this meeting should be sent to the Secretary of the Society at an early date.

THE January number (volume 12, number 2) of the *Annals of Mathematics* contains the following papers: "Rationality groups in prescribed domains," by S. EPSTEEN; "Envelopes of one parameter families of plane curves," by W. J. RISLEY and W. E. MACDONALD.

AT the meeting of the London mathematical society held on February 9 the following papers were read: By E. CUNNINGHAM, "The application of the mathematical theory of relativity to the electron theory of matter"; by G. B. MATHEWS and W. E. H. BERWICK, "The reduction of arithmetic binary forms which have a negative discriminant"; by H. BATEMAN, "Certain vectors associated with an electromagnetic field and the reflection of light at the surface of a perfect conductor."

THE royal academy of sciences of Turin announces the following conditions for the Vallauri prize: "A prize will be awarded by the Academy, without distinction of nationality, to the author, who, in the interval from January 1, 1915, to December 31, 1918, shall have published the most extensive and celebrated work in the domain of the physical sciences, the word being understood in its broadest sense. The amount of the prize is twenty-six thousand lire, to be paid one year after being awarded. Members of the Academy may not compete. No notice will be taken of memoirs in manuscript."

THE royal academy of sciences of Bologna announces the following prize problem: "To investigate by critical and historical methods the organic development of the theory of elliptic functions and the various points of view under which the theory has been considered from the end of the eighteenth century until the present. Indicate the influence which these various views have had upon other branches of analysis." Competing memoirs should be plainly written in Italian and sent, under the usual conditions, to the secretary of the academy before December 31, 1912.

THE Dutch scientific society announces the following prize problems, the award to be made January 1, 1912:

1. It is required to determine the prime numbers p which satisfy the congruence $g^{p-1} - 1 \equiv 0 \pmod{p^a}$, in which g and a are given integers larger than unity.

2. The society desires an experimental and theoretical study of the phenomena of critical opalescence, either in gases or in mixed liquids, and the particularities in the characteristic equation which can be attributed to the same causes as critical opalescence.

Competing memoirs should be sent, under the usual conditions, to the secretary, Dr. J. P. Lotsy, in Harlem. The prize is either a gold medal or a cash award of 150 florins, in the option of the successful competitor. In the case of a memoir of exceptional value, an extra sum of 150 florins may also be awarded.

THE Macmillan Company announces the following work as in press, to appear in a few weeks: Fundamental Concepts of Algebra and Geometry, by Professor J. W. YOUNG, of the University of Kansas.

THE next annual meeting of the British association for the advancement of science, will be held at Portsmouth, beginning August 30, under the presidency of Sir WILLIAM RAMSEY; Professor H. H. TURNER is chairman of the section of mathematical and physical sciences.

BEGINNING with volume 13 (1911), the *Enseignement Mathématique* will be published jointly by Gauthier-Villars in Paris, Georg and Co. in Geneva, and Teubner in Leipzig.

THE following doctorates in mathematics were conferred by the German universities during the academic year 1909–10:

Breslau.

DITTRICH, R. "Abstandsörter im Polarraume."

JURETZKA, E. "Die Entwickelung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen."

NEUMANN, NELLY. "Ueber das Flächennetz 2. Ordnung und seine korrelative Beziehung auf einen Strahlenbündel."

Erlangen.

BALDUS, R. "Über Strahlensysteme, welche unendlich viele Regelflächen 2. Grades enthalten."

GERSTENMEIER, C. "Beiträge zur Theorie der linearen Differentialgleichungen mit 4 und 5 singulären Stellen."

GEUS, A. "Die eindeutigen Transformationen der ebenen Kurve dritter Ordnung in sich, invarianten- und funktionentheoretisch behandelt."

Giessen.

WEHRHEIM, H. "Über das kombinatorische Produkt dreier Kollineationen in der Ebene."

Göttingen.

BIEBERBACH, L. "Zur Theorie der automorphen Funktionen."

COURANT, R. "Ueber die Anwendung des Dirichletschen Prinzipes auf die Probleme der konformen Abbildung."

FREUNDLICH, E. "Analytische Funktionen mit beliebig vorgeschriebenem unendlich-blättrigem Existenzbereiche."

JAEGER, M. "Graphische Integrationen in der Hydrodynamik."

KAHN, GRETE." "Eine allgemeine Methode zur Untersuchung der Gestalten algebraischer Kurven."

LÖBENSTEIN, KLARA. "Ueber den Satz, dass eine ebene, algebraische Kurve 6. Ordnung mit 11 sich einander ausschliessenden Ovalen nicht existiert."

WINK, A. "Über die Diskontinuitätsbereiche der Gruppen aus linearen nicht infinitesimalen Substitutionen."

Greifswald.

BEYER, T. "Die Integration der simultanen linearen Differentialgleichungen mit konstanten Koeffizienten."

Halle.

LEHMANN, P. "Beiträge zur Theorie der Darstellung der stetigen Funktionen durch Reihen von ganzen rationalen Funktionen."

ROESER, E. "Die Verfolgungskurve auf der Kugel."

Heidelberg.

CARLEBACH, J. "Lewi ben Gerson als Mathematiker."

Kiel.

KOCH, W. "Beiträge zur affinen Geometrie der Flächen zweiten Grades."

Königsberg.

GAEDECKE, W. "Die inversen Flächen der Mittelpunktsflächen 2. Ordnung."

SCHIMANSKI, E. "Die algebraischen Invarianten der projektiven Gruppen der Ebene und die geometrische Charakterisierung dieser Gruppen."

Leipzig.

ALBERTI, H. "Die Grundlagen des Systems Spinozas in Lichte der kritischen Philosophie und der modernen Mathematik."

FEYER, W. "Über die Höldersche Funktion

$$Fu = e^u \cdot \prod_{n=-\infty}^{+\infty}{}^{1} \{(1 - u/n)^n \cdot e^{u+\frac{1}{2}u^2/n}\}$$

und einige verwandte Transcendente."

München.

ASHTON, C. H. "Die Heineschen O-Funktionen und ihre Anwendungen auf die elliptischen Funktionen."

LOEHRL, A. "Über konforme und äquilonge Transformationen im Raum. Ein Beitrag zur Geometrie der Kugeln."

ROSENTHAL, A. "Untersuchungen über gleichflächige Polyeder."

Münster.

FERRARI, F. "Die geometrische Lösung der Aufgaben dritten und vierten Grades mittels des Lineals und einer festen Kurve dritter Ordnung mit Rückkehrpunkt oder reellem Doppelpunkte."

Rostock.

DÜKER, W. "Über Beziehungen der Strahlenkomplexe zweiten Grades zu den Flächen zweiter Ordnung."

NADLER, C. "Über den Zusammenhang der Raumkurve vierter Ordnung erster Spezies mit ihrem Polartetraeder."

WOLFF, H. "Behandlung des Vorganges, dass eine ebene elektromagnetische Welle, die auf die ebene Oberfläche eines Körpers, insbesondere eines Leiters auftrifft, von diesem reflectiert wird, auf Grund der Maxwellschen Gleichungen unter ausführlichen Eingehen auf die Art der stattfindenden Energiefortpflanzung."

Strassburg.

BRAUN, WANDA. "Bestimmung der Körperdiskriminante in einem kubischen Zahlkörper."

BURGWEDEL, R. "Über die Eulerschen und Gaussschen Methoden der Primzahlbestimmung."

GIROD, L. "Das sphärische Analogen der Hypocykloïdenbewegung des Cardanus und sein Zusammenhang mit der Theorie eines verallgemeinerten Hooke'schen Gelenkes."

KIEFER, A. "Die Einführung der homogenen Koordinaten durch K. W. Feuerbach."

PLATE, H. "Punktausgleichung und Fehlerbestimmung nach graphischen Methoden in ihrer Anwendung auf Ortsbestimmung durch Standlinien."

STAMPFLI, O. "Der Zweiteilungskörper der elliptischen Funktionen."

Tübingen.

BLUM, F. "Die infinitesimale Biegung von Flächen bei vollständiger Starrheit eines Kurvensystems."

Würzburg.

GRÄBNER, G. "Algebraische Bertrand-Kurven und algebraische Kurven konstanter Torsion."

DR. W. SIERPINSKI, of the University of Lemberg, has been promoted to an associate professorship of mathematics.

PROFESSOR A. SOMMERFELD, of the University of Munich, has been elected a member of the Bavarian academy of sciences.

PROFESSOR P. STÄCKEL, of the technical school at Carlsruhe, has been elected to membership in the Heidelberg academy of sciences.

PROFESSOR M. PASCH, of the University of Giessen, will retire at the end of the present semester.

PROFESSOR W. F. MEYER, of the University of Königsberg, has received the title of Geheimer Regierungsrat.

PROFESSOR L. SCHLESINGER, of the University of Klausenburg, has accepted the professorship of mathematics at the University of Budapest.

PROFESSOR H. W. KUHN, of the University of Ohio, has been granted leave of absence during the second half of the present academic year, to study in Europe.

DR. ANNA J. PELL has been appointed instructor in mathematics at Mount Holyoke College.

PROFESSOR A. W. PHILLIPS, of Yale University, will retire from active service at the close of the present academic year. Professor Phillips has taught in Yale College since 1876, becoming professor of mathematics in 1891. Since 1895 he has been the dean of the graduate school.

MR. C. J. WHITE, professor of mathematics at Harvard University from 1885 to 1894, has been appointed professor emeritus.

PROFESSOR CHARLES MÉRAY, of the University of Dijon, died February 7, at the age of 75 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BARBETTE (E.). Le dernier théorème de Fermat. Liège, Gnuse. 8vo. 19 pp.

BERZOLARI (L.). Geometria analitica, I: Il metodo delle coordinate. (Collection Manuali Hoepli.) Milan, Hoepli. 16mo. 409 pp. L. 3.00

FABRY (E.). Théorie des séries à termes constants. Paris, Hermann, 1910. 8vo. 200 pp. Fr. 6.50

FESTSKRIFT, H. G. ZEUTHEN. Fra venner og elever i anledning af hans 70 aars födselsdag. Kjöbenhavn, Höst, 1910. 8vo. 156 pp.

FREUNDLICH (E.). Analytische Funktionen mit beliebig vorgeschriebenen unendlich-blättrigen Existenzbereichen. (Diss.) Göttingen, 1910. 8vo. 40 pp.

FRICKE (R.) und KLEIN (F.). Vorlesungen über die Theorie der automorphen Funktionen. 2ter Band: Die funktionentheoretischen Ausführungen und die Anwendungen. 2te Lieferung: Kontinuitätsbetrachtungen im Gebiete der Hauptkreisgruppen. Leipzig, Teubner, 1911. 8vo. pp. 283–438. M. 7.00

HUYGENS (C.). Tome XII: Travaux de mathématiques pures. Paris, Gauthier-Villars. Fr. 20.00

JÜTTE (O.) Die Schmiegungskugel einer Flächenkurve. (Diss.) Halle, 1910. 8vo. 8+143 pp.

KLEIN (F.). See FRICKE (R.).

KNESER (A.). Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Vorlesungen an der Universität zu Breslau gehalten. Braunschweig, Vieweg, 1911. 8vo. 8+243 pp. M. 7.00

LEHMANN (F. J.). Harmonic analysis. Oberlin, Comings, 1910. 8vo. 9+156 pp. $1.50

MEDUGNO (M.). Della trasformazione birazionale [2–4]. Napoli, Accademia delle Scienze, 1910. 8vo. 188 pp.

MERKENS (E.). Ueber gewisse räumliche Punktmengen, die sich als stetige Flächen auffassen lassen. (Diss.) Königsberg, 1910. 8vo. 86 pp.

SPANO (D.). Sulla identità degli sviluppi in serie del Wronski e del Lagrange. Napoli, Accademia delle Scienze, 1910. 8vo. 10 pp.

VIVANTI (G.). Lezioni di analisi infinitesimale. Puntata I. Pavia, Mattei, 1910. 8vo. 304 pp. L. 15.00

VOIGT (A.). Theorie der Zahlenreihen und der Reihengleichungen. Leipzig, Göschen, 1911. 8vo. 8+133 pp. M. 4.00

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

ANZLINGER. Algebraische Aufgaben mit vollständiger Lösung durch einfache Schlüsse. Buhl, Baden, Konkordia A.-G.

ARMENI (A.). Ricreazioni geometrico-floreali policrome. Milano, Vallardi, 1910. 4to.

AUSSANT-CARA (P.). Sulla discussione dei problemi riducibili al secondo grado, con varie applicazioni. Livorno, Giusti, 1910. 8vo. 52 pp. L. 1.30

CLARIN (J.). Cours de mathématiques générales. Tome I: Algèbre, géométrie analytique, calcul différentiel. Lille, Janny. 4to. 516 pp.

COLIN (C.) et GIROD (J.). Cours de géométrie, conforme aux programmes officiels du 26 juillet 1909. 2e année, renfermant les compléments pour toutes les sections. Paris, Alcan, 1910. 18mo. 6+240 pp. Fr. 2.50

ENRIQUES (F.) e AMALDI (U.). Nozioni di geometria, ad uso dei ginnasi inferiori. Bologna, Zanichelli, 1910. 16mo. 158 pp. L. 1.50

FÄRBER (C.). Grundlehren der Mathematik für Studierende und Lehrer. 1ter Teil, 1ter Band: Arithmetik. Leipzig, Teubner, 1911. 8vo. 15+410 pp. M. 9.00

FOUBERT (E.). See ROLLET (P).

FRATTINI (G.). Geometria intuitiva, per uso delle scuole complementari e del ginnasio inferiore. 3a edizione. Torino, Paravia, 1910. 16mo. 117 pp. L. 1.00

GAZZANIGA (P.). See VERONESE (G.).

GIBSON (G. A.) and PINKERTON (P.). Elements of analytical geometry. London, Macmillan. 8vo. 498 pp. 7s 6d.

GIROD (J.). See COLIN (C.).

HESS (A.). Trigonometrie für Maschinenbauer und Elektrotechniker. Berlin, Springer, 1911. 8vo. 7+128 pp. Cloth. M. 2.80

KAMBLY und LANGGUTH. Arithmetik und Algebra, neu bearbeitet von A. Thaer. Ausgabe C: Für Realschulen. Breslau, Hirt, 1911. 8vo. 96 pp. M. 1.25

LEPOIVRE (G.) et POIRSON (A.). Cours de géométrie théorique et pratique à l'usage des élèves des écoles pratiques d'industrie, des écoles professionnelles et des écoles primaires supérieures. 2e édition, revue et augmentée. Lille, Janny, 1910. 8vo. 6+213 pp.

MACDOUGALL (W.). Common sense arithmetic. London, Collins. 8vo. 112 pp. 1s.

NISSEN (F. J.). A description of the details and results by the use of the Nissen number system. Port Clinton, Nissen, 1911. 8vo. 97 pp. $5.00

OTTO (F.), PETRI (W.) und ZIEGLER (I.). Mathematik für Lyzeen. In 3 Teilen. 1ter Teil: Lehrstoff der Klassen I und II. Leipzig, Hirt, 1910. 8vo. 7+299 pp. Cloth. M. 4.00

PAPE (L. A. J.). Courte étude d'instruments et de procédés géométriques (personnels). Paris, Pape, 1910. 8vo. 4pp.

PAPELIER (G.). Précis d'algèbre, d'analyse et de trigonométrie, à l'usage des élèves de mathématiques spéciales. 3e édition. Paris, Vuibert et Nony, 8vo. 452 pp.

PETRI (W.). See OTTO (F.).

PINKERTON (P.). See GIBSON (G. A.).

POIRSON (A.). See LEPOIVRE (G.).

REALE (E.) e RICOZZI (F.). Trattato di disegno geometrico lineare diviso in tre parti, per le scuole medie. Seconda edizione, riveduta ed ampliata. Parte I. Napoli, Casella, 1910. 8vo. 68 pp. L. 1.25

RICOZZI (F.). See REALE (E.).

ROLLET (P.) et FOUBERT (E.). Cours d'algèbre conforme aux programmes des 26 juillet et 28 août 1909. 8e édition, entièrement refondue. Paris, Alcan, 1910. 18mo. 8+444 pp. Fr. 3.00

ROTHROCK (D. A.). Elements of plane and spherical trigonometry, without tables. New York, Macmillan, 1911. 8vo. 11+147 pp. $1.10

——. Logarithmic, trigonometric and other tables. New York, Macmillan, 1911. 8vo. 14+99 pp. $0.60

RUTLAND (N.). Practische Anleitung zum gründlichen Unterrichte in der Buchstabenrechnung. 1ster Teil: Die allgemeine Arithmetik und Algebra. 8te verbesserte Auflage. Bonn, Cohen, 1911. 8vo. 12+440 pp. M. 6.00

SANDERS (H. J.). Elementary graphs, algebra. London, Collins. 8vo. 176 pp. 8d.

——. Elementary graphs, arithmetic. London, Collins. 8vo. 88 pp. 8d.

——. Intermediate arithmetic and geometry. London, Collins. 8vo. 176 pp. 1s. 6d.

SCOTO (G.). Nozioni pratiche di geometria: libro di testo per le tre classi del ginnasio inferiore. 4ª edizione. Palermo, Sandron, 1910. 16mo. 182 pp. L. 2.00

SERRET (J. A.). Trattato di trigonometria. Versione autorizzata dall' autore, con aggiunte per L. Fenoglio. (Collezione Paravia.) Terza edizione. Torino, Paravia, 1911. 8vo. 218 pp. L. 3.00

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